LOCAL Lp-INVERSE THEOREMS OF A GENERAL ORDER FOR LINEAR COMBINATIONS OF CERTAIN LINEAR POSITIVE OPERATORS

By

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PERI SARWESWARA AVADHANI

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DECEMBER, 1982

DEDICATED

TO

MY PARENTS

CERTIFICATE

Certified that the work presented in this thesis entitled 'LOCAL L_p-INVERSE THEOREMS OF A GENERAL ORDER FOR LINEAR COMBINATIONS OF CERTAIN LINEAR POSITIVE OPERATORS' by Shree Peri Sarweswara Avadhani has been carried out under my supervision and that this has not been submitted elsewhere for a degree or diploma.

December, 1982 — *(R.K.S. RATHORE)*

CERTIFICATE

Certified that the work presented in this thesis entitled 'LOCAL L_p-INVERSE THEOREMS OF A GENERAL ORDER FOR LINEAR COMBINATIONS OF CERTAIN LINEAR POSITIVE OPERATORS' by Shree Peri Sarweswara Avadhani has been carried out under my supervision and that this has not been submitted elsewhere for a degree or diploma.

December, 1982

(R.K.S. RATHORE)

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Avadhani

(Peri Sarweswara Avadhani)

CONTENTS

SYNOPSIS

Local inverse theorems in the approximation by linear operators have been of considerable interest during the past decade. The initial problems considered in this connection concerned approximation by sequences of linear positive operators. Subsequently, their linear combinations, which could provide a faster degree of approximation, were taken up. Local inverse theorems for several sequences of linear positive operators and their linear combinations have been obtained during this period. The orders of approximation studied in these local inverse theorems have been $n^{-\alpha}$, α a positive number.

A representative inverse result which has been shown to be valid for several sequences of linear positive operators is: $||L_n(f;t) - f(t)||_{L_p[a,b]} = O(n^{-\alpha/2})$ implies that $\omega_2(f,h,p,[a_1,b_1]) = O(h^{\alpha})$, where $a < a_1 < b_1 < b$. A similar result for linear combinations $L_n(\cdot,k,t)$ of the operators L_n defined by

$$L_n(\cdot,k,t) = \sum_{j=0}^{k} \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{d_j}{d_j-d_i} L_{d_j n}(\cdot,t),$$

shown to be valid for several sequences $\{L_n\}$ of linear positive operators is: $||L_n(f,k,t)-f(t)||_{L_p[a,b]} = O(n^{-\alpha/2})$ implies that $\omega_{2k+2}(f,h,p,[a_1,b_1]) = O(h^{\alpha})$, where $a < a_1 < b_1 < b$.

It may also be noted that the classical Bernstein inverse theorems too concern with the order $O(n^{-\alpha})$ of approximation by trigonometric polynomials of best approximation, for instance.

The present thesis considers local L_p-inverse theorems for linear combinations of linear positive operators of a much more general order $\varphi(n^{-1/2})$, where the order function φ is subject only to the following constraints: φ is a positive function on $(0,c]$, for some $c > 0$, such that

1. $$\sup_{t \in (0,c]} \frac{\varphi(t)}{\varphi(th)} \leq K_\varphi(h),$$

where $K_\varphi(h)$ is a positive function defined for $h \in (0,1)$.

2. $\sup_{h \geq \delta} K_\varphi(h) < \infty$, for all $\delta \in (0,1)$

and

3. $h^r K_\varphi(h) \to 0$ as $h \to 0$, for an appropriate positive integer r. Let Φ_r denote the collection of functions φ satisfying the above properties.

The function $\varphi(t) = t^\alpha (\alpha > 0)$, which corresponds to the order $n^{-\alpha} (= \varphi(n^{-1}))$ belongs to φ_r, for any integer $r > \alpha$.

Some non-trivial examples of $\varphi \in \Phi_r$ are as follows:

1. For all $t \in (0,c]$,

$$\varphi(t) = t^\alpha (\log^p(\tfrac{1}{t}))^q \in \Phi_r,$$

where p is a positive integer, $q > 0$, $0 < \alpha < r$, $c > 0$ with

$\log^p 1/c > 0$ and $\log^p t$ denotes the function $\log(\log\ldots(\log t))$ (p times).

2. For all $t \in (0,c]$,

$$\varphi(t) = \frac{t^\alpha}{(\log^p(\frac{1}{t}))^q} \in \Phi_r ,$$

where p is a positive integer and $q > 0$, $0 < \alpha < r$ and $c > 0$ such that $\log^p \frac{1}{c} > 0$.

3. Let $a < b$ be extended real numbers and $r > k \in \mathbb{N}$ and $f \in L_p[a,b]$ $(1 \leq p < \infty)$ be a function which is not a polynomial of degree $\leq k-1$. Then

$$\varphi(t) = \omega_k(f,t,p,[a,b]) \in \Phi_r ,$$

where $t \in (0,c]$, $c < \frac{b-a}{2k}$.

The representative $O(\varphi)$-inverse result which, in this thesis, has been shown to be valid for several sequences of linear combinations of linear positive operators reads:
Let $\varphi \in \Phi_{2k+2}$. Then

$$||L_n(f,k,t) - f(t)||_{L_p[a,b]} = O(\varphi(n^{-1/2}))$$

implies that

$$\omega_{2k+2}(f,t,p,[a_1,b_1]) = O(\varphi(t)),$$

where $a < a_1 < b_1 < b$.

The sequences and classes of operators which have been considered in the thesis are:

(a) the operators U_n of Sikkema and Rathore defined for $f \in L_p(-\infty,\infty)$ by

$$U_n(f;x) = \frac{1}{\alpha(n)} \int_{-\infty}^{\infty} f(t)\ \beta^n(x-t)\,dt,$$

where

$$\alpha(n) = \int_{-\infty}^{\infty} \beta^n(t)\,dt$$

and $\beta \in L_1(-\infty,\infty)$ is a bell-shaped function satisfying: $\beta(0) = 1$, β infinitely differentiable in a neighbourhood of origin, $\beta''(0) \neq 0$ and $\sup\ \{\ \beta(t) : |t| \geq \delta\ \} < 1$ for every $\delta > 0$.

(b) The Bernstein-Kantorovitch polynomials P_n defined for $f \in L_p[0,1]$ by

$$P_n(f;x) = \sum_{k=0}^{n} \binom{n}{k}\ x^k(1-x)^{n-k}(n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt.$$

(c) The generalized Müller's operators T_λ defined for $f \in L_p(0,\infty)$ as,

$$T_\lambda(f;x) = \frac{x^{\alpha-1}}{a(\lambda)} \int_0^{\infty} u^{-\alpha}\ G^\lambda(x/u)\ f(u)\,du,$$

where

$$a(\lambda) = \int_0^{\infty} u^{\alpha-2}\ G^\lambda(u),$$

α is a real number, G is a measurable function differentiable in a neighbourhood of the point '1', $\sup\ \{G(u) : |u-1| \geq \delta\} < G(1)$ for all $\delta > 0$ and $(u^{-a}+u^b)\ G(u)$ being bounded on $(0,\infty)$ for some $a,b > 0$.

(d) The regular exponential type operators S_n defined for $f \in L_p[A,B]$ (A,B distinct extended real numbers) by

$$S_n(f;t) = \int_A^B W(n,t,u)\, f(u)\,du,$$

where the kernel $W(n,t,u)$ satisfies:

(i) $\quad \int_A^B W(n,t,u)\,du = 1$

(ii) $\quad$ For $u \in (A,B)$, $\int_A^B W(n,t,u)\,dt = a(n) \to 1 \ (n \to \infty)$,

$a(n)$ being a rational function of n,

(iii) $\quad \dfrac{\partial W}{\partial t} = \dfrac{n}{p(t)}(u-t)W, \quad u,t \in (A,B)$

so that

$$\frac{d^k}{dt^k} S_n(f;t) = \int_A^B \frac{\partial^k}{\partial t^k} W(n,t,u)\, f(u)\,du$$

and

(iv) $\quad t^m p(t)\, W(n,t,u) \to 0$ as $t \to A,B$

for all sufficiently large values of n, and

(e) the generalized Jackson operators

$$L_{n,p}(f;x) = \frac{1}{A_{n,p}} \int_{-\pi}^{\pi} \left(\frac{\sin n\frac{(t-u)}{2}}{\sin \frac{t-u}{2}}\right)^{2p} f(t)\,dt$$

with $\quad A_{n,p} = \displaystyle\int_{-\pi}^{\pi} \left(\frac{\sin \frac{nt}{2}}{\sin t/2}\right)^{2p} dt,$

where p is a positive integer.

Apart from considering the linear combinations of the above operators we have also considered interpolatory modifications

$P_{n,m}$ and $S_{n,m}$ of the operators P_n and S_n, respectively, defined by

$$P_{n,m}(f;t) = (n+1)\sum_{\nu=0}^{n}\sum_{j=0}^{m}\frac{n^{j/2}}{j!}\,p_{n\nu}(t)\,\times$$

$$\times\int_{\frac{\nu}{n+1}}^{\frac{\nu+1}{n+1}}\left(\prod_{i=0}^{j-1}\left(t-u-\frac{i}{n^{1/2}}\right)\right)\Delta^{j}_{n^{-1/2}}f(u)\,du,$$

where $\prod_{i=0}^{j-1}(t-u-i/n^{1/2})$ for $j = 0$ is interpreted as 1, $f(u)$ is regarded as zero when $u > 1$ and

$$p_{n\nu}(t) = \binom{n}{\nu}\,t^{\nu}\,(1-t)^{n-\nu}$$

and

$$S_{n,m}(f;t) = \int_{A}^{B}\frac{W(n,t,u)}{1+|u-t|^{m_0+2}}\,\times$$

$$\times\left\{\sum_{j=0}^{m}\frac{n^{j/2}}{j!}\left(\prod_{i=0}^{j-1}\left(t-u-\frac{i}{n^{1/2}}\right)\right)\Delta^{j}_{n^{-1/2}}f(u)\right\}du$$

where $m \leq m_0$ and $W(n,t,u)$ is the kernel of the operators S_n.

In addition to obtaining $O(\varphi)$-inverse theorems for the above operators, we have also introduced and obtained $o(\varphi)$-inverse theorems. A $o(\varphi)$-inverse result for linear combinations $L_n(\cdot,k,t)$ reads as:

Let $\varphi \in \Phi_{2k+2}$. Then

$$\|L_n(f,k,t) - f(t)\|_{L_p[a,b]} = o(\varphi(n^{-1/2}))$$

implies that

$$\omega_{2k+2}(f,h,p,[a_1,b_1]) = o(\varphi(h)),$$

where $a < a_1 < b_1 < b$.

This has been proved to be valid for linear combinations and interpolatory modifications defined above.

The main ingradients of the proofs of earlier known inverse theorems have been Bernstein type inequalities and moments of the operators, a lemma of Berens and Lorentz and Peetre's K-functionals or Steklov type means. For obtaining φ-inverse theorems in the thesis, two generalisations of the Berens-Lorentz lemma and two generalisations of the result :

$$\omega_{k+1}(f,t,p,[a,b]) = O(t^{\alpha})$$

implies that

$$\omega_{k}(f,t,p,[a,b]) = O(t^{\alpha}),$$

whenever $\alpha < k$, have been obtained and are found to be crucial. The technique developed for φ-inverse theorems seems to be of a general applicability also with respect to sup-norm and several more sequences of linear positive operators.

The thesis is devided into six chapters. Chapter 0 is introductory in nature and contains some basic definitions, notations and a brief survey of the chapterwise contents of the thesis. In Chapter I we obtain φ-inverse theorems for the linear combinations $U_n(\cdot,k,t)$. In Chapter II

φ-inverse theorems for linear combinations $P_n(\cdot,k,t)$ and interpolatory modifications $P_{n,m}(\cdot,t)$ have been obtained. Chapter III is a study of φ-inverse theorems of linear combinations of generalised Muller's operators T_λ. In Chapter IV φ-inverse theorems for linear combinations $S_n(\cdot,k,t)$ and interpolatory modifications $S_{n,m}(\cdot,t)$ of regular exponential type operators have been obtained. Finally, in Chapter V, φ-inverse theorems have been obtained for linear combinations $L_{n,p}(\cdot,k,t)$ of the generalised Jackson operators.

CHAPTER 0

INTRODUCTION AND CONTENTS OF THE THESIS

0.1 Introduction.

Ditzian and May [22] proved the following theorem : Let $0 < a < a_1 < b_1 < b < 1$, $f \in L_p[0,1]$ and $||P_n f - f||_{L_p[a,b]} \leq K n^{-\alpha/2}$ for some $0 < \alpha < 2$, then, for all $h \leq \frac{1}{2} \min(b-b_1, a_1-a)$,
$\omega_2(f,h,p,[a_1,b_1]) = \sup_{|\tau|<h} ||\Delta_\tau^2 f||_{L_p[a_1,b_1]} \leq Mh^\alpha$, where

$$P_n(f,t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} (n+1) \int_{k/n+1}^{(k+1)/n+1} f(u)\, du,$$

is the nth Bernstein-Kantorovitch polynomial of f. Sinha [61], subsequently, generalised this result to the linear combinations

$$P_n(f,k,t) = \sum_{j=0}^{k} \prod_{\substack{i=0 \\ i\neq j}}^{k} \frac{d_j}{d_j - d_i} P_{d_j n}(f;t)$$

of the Bernstein-Kantorovitch polynomials as follows : Let $0 < a < a_1 < b_1 < b < 1$, $f \in L_p[0,1]$ and $||P_n(f,k,t) - f(t)||_{L_p[a,b]} = O(n^{-\alpha/2})$ for $0 < \alpha < 2k+2$, then

$$\omega_{2k+2}(f,h,p,[a_1,b_1]) = \sup_{|\tau|<h} ||\Delta_\tau^{2k+2} f||_{L_p[a_1,b_1]} = O(h^\alpha).$$

The work of the present thesis arose out of the following question to which we addressed ourselves to begin with : Suppose $||P_n(f,k,t) - f(t)||_{L_p[a,b]}$ or $||P_n(f;t) - f(t)||_{L_p[a,b]}$ is of $O(n^{-\alpha/2} \log n)$ or $||P_n(f,k,t) - f(t)||_{L_p[a,b]}$ or

$||P_n(f;t)-f(t)||_{L_p[a,b]}$ is of $O(n^{-\alpha/2}/\log n)$, then what can we say about the smoothness of the function f? The answer to this question turned out to be : $\omega_{2k+2}(f,t,p,[a_1,b_1]) = O(t^{\alpha}\log(1/t))$, $\omega_2(f,t,p,[a_1,b_1]) = O(t^{\alpha}\log(1/t))$, $\omega_{2k+2}(f,t,p,[a_1,b_1]) = O(t^{\alpha}/\log(1/t))$ and $\omega_2(f,t,p,[a_1,b_1]) = O(t^{\alpha}/\log(1/t))$, respectively. The form of the above answer was quite suggestive and it led us to a consideration of the more general problem of obtaining inverse theorems of a general order. The present thesis is the result of this undertaking.

The main ingredients of the proofs of earlier theorems were Bernstein type inequalities and moments of the operators, a lemma of Berens-Lorentz and Peetre's K-functionals or Steklov type means. We found that two generalisations of Berens-Lorentz lemma and two generalisations of the result : $\omega_{k+1}(f,t,p,[a,b]) = O(t^{\alpha})$ $(t \to 0)$ implies that $\omega_k(f,t,p,[a,b]) = O(t^{\alpha})$, whenever $\alpha < k$, turn out to be sufficient to provide us with a machinery by which the problem of a general order inverse theorem could be solved. In the present thesis, we have illustrated this for five well known classes or sequences of operators. However, it may be easily seen that the technic is of quite a general applicability (also with respect to sup-norm case). We have restricted ourselves only to the local approximation theorems. This is because we feel that approximation by linear operators seems specially tailored for taking advantage of local smoothness of

functions, as may be concluded from discussion of the subsequent section.

0.2 Local approximation.

Consider $f(x) = |x - \frac{1}{2}|^{1/10}$, $x \in [0,1]$. The polynomial P_n of best approximation of degree n to this function on $[0,1]$ in sup-norm, has precise order $n^{-1/10}$ of approximation. In view of the Chebyshev equi-oscillation property it may be concluded that the same order of approximation persists even on an interval $[.5+\delta,1]$ for any fixed $\delta \in (0,.5)$ (Infact, in the present case, $\sup\{|f(x)-P_n(x)| : x \in [0,1]\} = |f(1)-P_n(1)|\}$. However, it may be noted that the function is badly behaved only in a neighbourhood of $x = 1/2$ and that on $[.5+\delta,1]$ it is a nice analytic function. Thus the polynomial of best approximation does not necessarily take advantage of local nice behaviour of functions. On the other hand, consider $B_n(f;t)$, the nth Bernstein polynomial of the same function. The order of approximation on $[0,1]$ by $B_n(f;t)$ is $O(n^{-\frac{1}{20}})$, which is poor in comparison with the order of approximation by P_n's. However, on $[.5+\delta,1]$ with δ as above, the order of approximation by B_n's is $O(n^{-1})$ and, therefore, B_n's indeed, take advantage of the local smoothness of functions in local approximation. Further, the linear combinations $\mathcal{L}_n^{[2k]}$ of B_n's defined by Butzer [14] approximate the function f on $[.5+\delta,1]$ with the order $n^{-(k+1)}$. This type of behaviour, i.e., providing a better approximation in the regions where the function is better behaved, is commonly

shared by several well known sequences of linear approximation operators. Some of the works concerned with the degree of (local/general) approximation by linear positive operators are those of Berens and Devore [9], Bojanic and Shisha [13], Hoeffding [26], Korovkin [30,31], Lorentz [35,36], May [40], Müller [45], Müller and Maier [46], Rathore [48,50] and Wood [68] and others.

0.3 Linear combinations of linear positive operators.

Among linear approximation operators, linear positive operators occupy a special place of interest (Korovkin [31]). However, the positivity of such operators places a severe constraint on their degree of approximation (Devore [19] and Korovkin [31]). To compensate for this several methods, e.g., using their linear combinations (Butzer [14], Rathore [48,49] May [39], Ditzian [20], Rathore and Agrawal [52], Rathore and Singh [53], Singh [60], Kunwar [32], Agrawal [1], Sinha [61] and Winslin [67] etc.), the linear combinations of iterates of linear positive operators (Miccheli [42], Agrawal [1], Bleimann, Junggeburth and Stark [12], Butzer and Berens [16,p105]), multiplying the kernels of linear positive operators by suitable factors so as to produce kernels with appropriate finite oscillations (Stark [62,63], Hoff [27]; see also Ditzian and Freud [23]) and interpolatory modifications recently considered by Sinha [61], etc. have been proposed in the literature. Most of the above methods have been studied

only with respect to the sup-norm. The study of linear combinations and interpolatory modifications of Sinha [61], however, is with respect to L_p-norms.

0.4 Order of approximation.

Approximation theorems concerned with the degree of approximation are mainly categorised as direct, inverse and saturation theorems. Indeed, a direct theorem tells the degree of approximation from a known smoothness property of a function, the inverse theorem infers the degree of smoothness from a given degree of approximation and a saturation theorem concerns with the fastest degree of approximation possible for a given method.

In this connection, most of the results particularly on inverse theorems for linear combinations etc. of linear positive operators, available in the literature, concern with the order $O(n^{-\alpha})$, α a real number, of approximation (Agrawal [1], Ditzian [21], Kunwar [32], May [39], Rathore [48] Rathore and Singh [53], Sinha [61] and Winslin [67]). In the classical part of approximation theory (concerned with polynomials and entire functions of exponential type etc. of best approximation) also, the Bernstein theorems (which probably are the first known inverse theorems) correspond to the very order $O(n^{-\alpha})$. Even for basic sequences and classes of linear positive operators the inverse theorems available are with respect to the order $O(n^{-\alpha})$ of approximation (e.g. in the works

of Becker [2,3] , Becker, Kucharski and Nessel [4] , Becker and Nessel [5 ,6 ,8] , Berens and Lorentz [10] , Butzer and Berens [16] , Butzer and Nessel [17] , Devore [19] , Ditzian [20], Ditzian and May [22] and **de Leeuw** [18] etc.).

The basic theme of this thesis is to extend the degree of approximation results (mainly, inverse theorems) to orders of approximation more general than $O(n^{-\alpha})$. The function which occurs in the general orders $O(\varphi(n^{-1/2}))$, with which we concern ourselves in the thesis, is only subject to the following restrictions :

$$\sup_{t\in(0,c]} \frac{\varphi(t)}{\varphi(th)} \leq K_\varphi(h), \text{ where for any } \delta > 0,$$

$$\sup_{1>h\geq\delta} K_\varphi(h) < \infty \quad \text{and for an appropriate positive integer } r.$$

$h^r K_\varphi(h) \to 0$ as $h \to 0$. We may note that $\varphi(t) = t^\alpha$ (which corresponds to approximation orders of type $n^{-\alpha}$) satisfies the above conditions for any integer $r > \alpha$. In view of this, the earlier known inverse theorems turn out to be very special cases of the results obtained in the thesis. Some other examples of functions φ with the above properties are

$$\varphi(t) = t^\alpha(\log^p (1/t))^q \qquad (\alpha < r),$$

$$\varphi(t) = t^\alpha/(\log^p 1/t)^q \qquad (\alpha < r)$$

and (a very general case)

$$\varphi(t) = \omega_k(t) \qquad (k < r),$$

where $\omega_k(t)$ is a kth modulus of continuity (with respect to

any of sup or integral L_p-norms) of some function on some interval.

In the thesis, in addition to $O(\varphi)$-inverse theorems, we also obtain inverse theorems of $o(\varphi)$-type. These have the flavour of the little-o Zygmund theorem as described in Natanson [47 , Theorem 3, p. 109] .

The approximation operators which we study with respect to the general order φ of approximation are linear combinations or interpolatory modifications of the following five sequences or classes of linear positive operators : (a) convolution operators U_n (Sikkema and Rathore [58]) on $(-\infty,\infty)$ generated by powers of bell-shaped functions as kernels, (b) Bernstein-Kantorovitch polynomials P_n (Kantorovitch [29]), (c) generalised Müller operators T_λ (Kunwar [32 ,33]) which are also generated by powers of a fixed function as kernels on $(0,\infty)$, (d) Regular exponential type operators S_n (Sinha [61] , Agrawal [1] and May [39]) and (e) generalised Jackson operators (Schurer [55,56] Rathore [50] and Lorentz [36]).

0.5 Some definitions and notations.

In this section we give some basic definitions and notations which we use throughout the thesis.

The symbols $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{N}^0$ denote the set of real numbers, positive real numbers, positive integers and non-negative integers, respectively. The proofs of various results in the

thesis extensively utilize constants which are independent of f (the function under consideration) and n (in U_n, P_n, S_n and $L_{n,p}$) or λ (in T_λ). Such constants will be denoted by M, $M_1, M_2, \ldots, M_1', \ldots$ etc. Thus, generally the presence of 'M' would implicitly mean that the constant is independent of f and n or λ.

Let $1 \leq p < \infty$. Then $L_p[a,b]$ is defined as the class of all complex valued measurable functions f (with the usual identification of functions equal a.e.) for which $\int_a^b |f(x)|^p dx < \infty$, where the integration is in the Lebesgue sense and the norm in $L_p[a,b]$ is defined by

$$||f||_{L_p[a,b]} = \left(\int_a^b |f(x)|^p \, dx\right)^{1/p}.$$

The space $L_\infty[a,b]$, similarly, consists of complex valued measurable functions which are essentially bounded on (a,b) and it is normed by

$$||f||_{L_\infty[a,b]} = \inf \{M : |f(x)| \leq M \text{ a.e. on } [a,b]\}.$$

The classes of all absolutely continuous functions over $[a,b]$ and functions of bounded variation over $[a,b]$ will be denoted, respectively, by A.C. $[a,b]$ and B.V. $[a,b]$. The class of k times continuously differentiable functions on $\mathbb{R}$ which have a compact support is denoted by C_0^k.

Next, we present some general definitions which are quite often used in the thesis. As per standard practice, numbers in the square brackets are those of pertinent references listed at the end of the thesis.

0.5.1 Newton's forward differences [25]. Let f be a complex valued function over $\mathbb{R}$ and $m \in \mathbb{N}$. Then, the mth forward difference of the function f at a point t, of step length δ, is defined as

$$\Delta_\delta^m f(t) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} f(t + j\delta).$$

Convetionally, we write $\Delta_\delta^0 f(t) = f(t)$.

0.5.2 Newton's interpolation polynomials [25]. If f is a complex valued function over $\mathbb{R}$ and $m \in \mathbb{N}$, then the Newton's interpolation polynomial $P_m(t)$ of degree m, which interpolates f at the equidistant points t_i, $i = 0,\ldots,m$, is given by

$$P_m(t) = \{ \sum_{j=0}^{m} \frac{1}{j!} \delta^j \prod_{i=0}^{j-1} (t-t_i) \Delta_\delta^j f(t_0) \},$$

where $\delta = t_{i+1} - t_i$, $i = 0,\ldots,m-1$ and $\prod_{i=0}^{j-1} (t-t_i) = 1$ if $j = 0$.

0.5.3 Hardy-Littlewood majorant [69]. For any $f \in L_p[a,b]$ $(1 < p < \infty)$, the Hardy-Littlewood majorant of f is defined as

$$H_f(x) = \sup_{\xi \neq x} \frac{1}{\xi - x} \int_x^\xi f(t)\,dt \quad (a \leq \xi \leq b).$$

0.5.4 Integral modulus of smoothness [65]. For any $f \in L_p[a,b]$ $(1 \leq p < \infty)$, the integral modulus of smoothness of f of order m is defined as

$$\omega_m(f,h,p,[a,b]) = \sup_{0 \leq \delta \leq h} ||\Delta_\delta^m f(t)||_{L_p[a,b-m\delta]}.$$

0.5.5 Steklov means [61] . Let $f \in L_p[a,b]$ $(1 \le p < \infty)$. Then, for $\eta > 0$, the Steklov means $f_{\eta,m}$ of order m, corresponding to f, is defined as follows :

$$f_{\eta,m}(t) = \eta^{-m} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \left(f(t) + (-1)^{m-1} \Delta^m_{\sum_{i=1}^{m} t_i} f(t)\right) dt_1 \ldots dt_m .$$

In the above integral we put $f(t) = 0$ if t does not belong to the domain (a,b) of definition of f.

0.5.6 Linear combinations [48] . Given a sequence $\{L_n(\cdot,t)\}$ of linear positive operators, we define their linear combinations $L_n(\cdot,k,t)$ as follows :

Let $d_0,\ldots,d_k$ be k+1 positive numbers such that $L_{d_j n}$'s are meaningful. Then

$$L_n(\cdot,k,t) = \frac{1}{\Delta_k} \begin{vmatrix} L_{d_0 n}(\cdot,t) & d_0^{-1} & \cdots & d_0^{-k} \\ L_{d_1 n}(\cdot,t) & d_1^{-1} & \cdots & d_1^{-k} \\ \vdots & & & \\ L_{d_k n}(\cdot,t) & d_k^{-1} & \cdots & d_k^{-k} \end{vmatrix}$$

where Δ_k is the determinent obtained from the displayed determinent after replacing the entries of the first column by 1.

These combinations can also be written as [39]

$$L_n(\cdot,k,t) = \sum_{j=0}^{k} C(j,k)\, L_{d_j n}(\cdot,t)$$

where the coefficients $C(j,k)$ are given by

$$C(j,k) = \begin{cases} \prod\limits_{\substack{i=0\\ i\neq j}}^{k} \dfrac{d_j}{d_j-d_i} & , \quad k \neq 0 \\ 1 & , \quad k = 0, \end{cases}$$

and they satisfy

$$\sum_{j=0}^{k} C(j,k)\; d_j^{-m} = \delta_{mo} ,$$

for $m = 1,2,\ldots,k$, where δ_{ij} is the Kronecker-delta equalling 1 if $i = j$ and 0 otherwise. (Thus, the linear combination $L_n(\cdot,k,t)$ is the kth degree Lagrange interpolation polynomial of $L_{x^{-1}n}(\cdot,t)$ (being regarded as function of x) interpolating at $x = d_o^{-1}, d_1^{-1}, \ldots, d_k^{-1}$ and evaluated at $x = 0$ [25].).

0.6 Certain basic results.

This section is a collection of some general results which will be used in the thesis.

We start with a lemma (Timan [65 , P.107]) which expresses an mth forward difference of a function in terms of an integral involving the mth derivative of the function.

Lemma 0.6.1. Let $f \in L_p[a,b]$ $(1 \leq p < \infty)$ and have an mth derivative such that

$f^{(m-1)} \in A.C.[a,b]$ and $f^{(m)} \in L_p[a,b]$. Then

$$\Delta_\delta^m f(t) = \int_0^\delta \cdots \int_0^\delta f^{(m)}(t + \sum_{i=1}^{m} y_i)\, dy_1 \ldots dy_m, \quad t \in [a, b-m\delta].$$

The next lemma [64, 69] gives a bound for the Hardy Littlewood majorant of a function.

Lemma 0.6.2. Let $f \in L_p[a,b]$ $(1 < p < \infty)$. Then the Hardy-Littlewood majorant H_f of f belongs to $L_p[a,b]$ and

$$||H_f||_{L_p[a,b]} \leq 2^{1/p} \frac{p}{p-1} ||f||_{L_p[a,b]}.$$

Now we state a standard inequality which qualitatively bounds an intermediate derivative in terms of the highest derivative and the function, in L_p-norms.

Lemma 0.6.3. Let $f \in L_p[a,b]$ $(1 \leq p < \infty)$ be such that $f^{(k)} \in$ A.C. $[a,b]$ and $f^{(k+1)} \in L_p[a,b]$. Then

$$||f^{(j)}||_{L_p[a,b]} \leq M_j \{||f^{(k+1)}||_{L_p[a,b]} + ||f||_{L_p[a,b]}\},$$

$j=1,2,\ldots,k$, where M_j's are certain constants depending only on j,k,p, a and b.

We shall also have occasions to use Riesz-Thorin's interpolation theorem [11, p. 2] which we state in

Lemma 0.6.4. Let $1 \leq p_o, q_o, p_1 (\neq p_o), q_1 (\neq q_o) \leq \infty$ and $T: L_{p_i}[a,b] \to L_{q_i}[c,d]$ be a bounded linear map with norm M_i $(i = 0,1)$. Then $T: L_p[a,b] \to L_q[c,d]$ with norm $M \leq M_o^{1-\theta} M_1^{\theta}$, where

$$\frac{1}{p} = \frac{1-\theta}{p_o} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_o} + \frac{\theta}{q_1} \quad \text{and} \quad 0 < \theta < 1.$$

Next we give some estimates about Steklov means in terms of integral modulii of smoothness of functions.

Lemma 0.6.5 [61] . Let $f \in L_p[a,b]$ $(1 \le p < \infty)$ and $[a_1,b_1] \subset (a,b)$. Then, for all sufficiently small $\eta > 0$, $f_{\eta,m}$ has derivatives upto order m, the (m-1)th derivative is absolutely continuous over $[a_1,b_1]$ and the mth derivative exists a.e. on $[a_1,b_1]$ and belongs to $L_p[a_1,b_1]$. Moreover there hold:

$$||f^{(r)}_{\eta,m}||_{L_p[a_1,b_1]} \le M_r \eta^{-r} \omega_r(f,\eta,p,[a,b]) \quad r = 1,2,\ldots,m,$$

$$||f-f_{\eta,m}||_{L_p[a_1,b_1]} \le M_{m+1} \omega_m(f,\eta,p\,[a,b]),$$

$$||f_{\eta,m}||_{L_p[a_1,b_1]} \le M_{m+2} ||f||_{L_p[a,b]},$$

$$||f^{(m)}_{\eta,m}||_{L_p[a_1,b_1]} \le M_{m+3} \eta^{-m} ||f||_{L_p[a,b]},$$

where the constants M_i's are independent of f and η.

An inequality relating a lower order integral modulus of smoothness with a higher order integral modulus of smoothness is given in

Lemma 0.6.6 [65] . Let $f \in L_p[a,b]$ $(1 \le p < \infty)$. Then

$$\omega_k(f,t,p,[a,b]) \le M_k t^k \{||f||_{L_p[a,b]} + \int_t^{\frac{b-a}{2k}} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}} du \},$$

where the constant M_k is independent of f.

We close this section by stating the well-known Minkowski inequality.

Lemma 0.6.7. If the function $f(x,y)$ is measurable on the rectangle $a \leq x \leq b$, $c \leq y \leq d$ and for some $q \geq 1$, we have

$$\int_c^d \{\int_a^b |f(x,y)|^q \, dx\}^{1/q} \, dy < \infty, \text{ then}$$

$$\{\int_a^b |\int_c^d f(x,y) \, dy|^q \, dx\}^{1/q} \leq \int_c^d \{\int_a^b |f(x,y)|^q \, dx\}^{1/q} \, dy.$$

0.7 A survey of the contents of the thesis

The thesis consists of five main chapters (excluding the present chapter 0). It is a study of local L_p-inverse theorems ($O(\varphi)$ and $o(\varphi)$) of a general order φ, for linear combinations of linear positive operators U_n, P_n, T_λ, S_n and $L_{n,p}$. Interpolatory modifications of the operators P_n and S_n are also considered. In each chapter we deal with one sequence or class of operators. A chapterwise summary of the thesis is as follows:

Chapter I: This is a study of the L_p-approximation ($1 \leq p < \infty$) by linear combinations $U_n(\cdot,k,t)$ of the operators U_n. Many well-known operators like Gauss-Weierstrass operators, De la Vallee Poussin Operators, Landau operators etc. are particular cases of these operators. Section 2 is a brief introduction about the general order φ of approximation under consideration. The results in this section are useful in proving all our φ-inverse theorems and contain a generalisation of a lemma of Berens and Lorentz [10] and another of Timan [65]. Section 3 contains some basic results

about the operators U_n and $U_n(\cdot,k,t)$. Lemma 1.3.5, in this section, is a localisation lemma, which implies that the degree of approximation by U_n or their linear combinations essentially depends on the behaviour of β in a neighbourhood of origin. In section 4 we prove a direct theorem for the linear combinations $U_n(\cdot,k,t)$. Sections 5 and 6 are devoted to our main results viz., the $O(\varphi)$ and $o(\varphi)$-inverse theorems. These are local in nature in the set up of contracting subintervals; so are the $O(\varphi)$ and $o(\varphi)$ inverse theorems in subsequent chapters.

<u>Chapter II</u>: In this chapter we study φ-inverse theorems for linear combinations $P_n(\cdot,k,t)$ and interpolatory modifications $P_{n,m}(\cdot,t)$ of Bernstein-Kantorovitch polynomials $P_n(\cdot,t)$. Section 1 gives a brief introduction about the operators and Section 2 consists of basic results about the operators $P_n(\cdot,t)$, $P_n(\cdot,k,t)$ and $P_{n,m}(\cdot,t)$. The $O(\varphi)$ and $o(\varphi)$-inverse theorems for linear combinations $P_n(\cdot,k,t)$ are obtained in Sections 3 and 4, respectively. Lastly, in Sections 5 and 6 we prove $O(\varphi)$ and $o(\varphi)$-inverse theorems for the operators $P_{n,m}(\cdot,t)$.

<u>Chapter III</u>. This chapter deals with linear combinations $T_\lambda(\cdot,k,t)$ of generalised Müller operators T_λ, which generalise several well-known operators including the Gamma operators of Müller [43] and Post-Widder [66] and modified Post-Widder operators [39]. The operators T_λ are defined

in Section 1. Basic results about T_λ, which are useful in later sections, are dealt with in Section 2 and in Section 3, we prove a direct theorem. To prove this, we first obtain a lemma which estimates the error in approximation in terms of derivatives of a function (lemma 3.3.2). Section 4 contains the $O(\varphi)$-inverse theorem for linear combinations $T_\lambda(\cdot,k,t)$ of the operators $T_\lambda(\cdot,t)$. Lastly, in Section 5, we prove $o(\varphi)$-inverse theorem for the operators $T_\lambda(\cdot,k,t)$.

Chapter IV. This chapter is a study of linear combinations $S_n(\cdot,k,t)$ and interpolatory modifications $S_{n,m}(\cdot,t)$ of regular exponential type operators. Section 1 is an introduction to the operators and Section 2 consists of basic results about the operators $S_n(\cdot,t)$, $S_{n,m}(\cdot,t)$ and $S_n(\cdot,k,t)$. Sections 3 and 4 are devoted, respectively, to the $O(\varphi)$ and $o(\varphi)$-inverse theorems for the operator $S_n(\cdot,k,t)$ and the last two sections, Sections 5 and 6, contain the $O(\varphi)$ and $o(\varphi)$-inverse theorems for the operators $S_{n,m}(\cdot,t)$.

Chapter V. This is the last chapter of the thesis dealing with the linear combinations $L_{n,p}(\cdot,k,t)$ of the generalised Jackson operators $L_{n,p}(\cdot,t)$ for functions which are 2π-periodic and belong to $L_q[-\pi,\pi]$ $(1 \leq q < \infty)$. The operators and their linear combinations are described in Section 1. Section 2 contains some basic results on the moments, the convergence and the L_p-boundedness of the operators. Section 3 is a direct theorem for $L_{n,p}(\cdot,k,t)$.

The last two sections(Sections 4 and 5) are devoted to $O(\varphi)$ and $o(\varphi)$-inverse theorems. It may be pointed out here that for $L_{n,p}(\cdot,k,t)$ we work with the order $\varphi(n^{-1})$ rather than the orders $\varphi(n^{-1/2})$ and $\varphi(\lambda^{-1/2})$ of earlier chapters. This is because the mth absolute moment of the operators $L_{n,p}(\cdot,t)$ is of order $O(n^{-m})$ whereas in the case of the operators of earlier chapters the mth absolute moments were of the order $O(n^{-m/2})$ and $O(\lambda^{-m/2})$. Lastly, we obtain global φ-inverse theorems for $L_{n,p}(\cdot,k,t)$ as a consequence of the local φ-inverse theorems.

CHAPTER I

φ-INVERSE THEOREMS FOR LINEAR COMBINATIONS OF OPERATORS U_n

1.1 Introduction.

In this chapter we study the linear combinations $U_n(\cdot,k,t)$ of a general class of convolution operators $U_n(\cdot,t)$ generated by powers of bell-shaped functions.

A function $\beta \in L_1(\mathbb{R})$ is said to be bell-shaped if

(i) $\beta(t) \geq 0$ for all $t \in \mathbb{R}$

(ii) $\beta(t)$ is continuous at $t = 0$ and $\beta(0) = 1$

(iii) $\sup\limits_{|t| \geq \delta} \beta(t) < 1$ for each $\delta > 0$.

The class of all bell-shaped functions will be denoted by B. By $B^{\infty,2}$ we mean the class of $\beta \in B$ which are infinitely differentiable in a neighbourhood of the origin and for which $\beta^{(2)}(0) \neq 0$. Also, for any $\delta > 0$, B_δ denotes the class of all $\beta \in B^{\infty,2}$ such that supp $\beta \subset (-\delta,\delta)$ and β is infinitely differentiable on $(-\delta,\delta)$.

With $\beta \in B$, the class $\{U_n : n \geq 1\}$ of linear positive convolution operators U_n is defined by

(1.1.1) $$U_n(f;t) = \frac{1}{\alpha(n)} \int_{-\infty}^{\infty} \beta^n(t-u)\, f(u)\, du$$

where

(1.1.2) $$\alpha(n) = \int_{-\infty}^{\infty} \beta^n(u)\, du.$$

Several approximation operators [58] can be obtained as particular cases of the operators U_n. Here are some examples :

1. If
$$\beta(t) = e^{-\frac{t^2}{2}}, \quad t \in \mathbb{R}$$
we get the Gauss-Weierstrass operators W_n defined by
$$W_n(f;t) = \frac{\sqrt{n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{n}{2}(u-t)^2} f(u)\, du$$

2. If
$$\beta(t) = \begin{cases} \cos^2 t/2, & t \in [-\pi,\pi] \\ 0, & t \in \mathbb{R} - [-\pi,\pi], \end{cases}$$
we get De la Vallée-Poussin operators V_n on the space $M_{2\pi}$ of all 2π-periodic, bounded, complex-valued measurable functions on $\mathbb{R}$, defined by
$$V_n(f;t) = \frac{(2n)!!}{2\pi(2n-1)!!} \int_{-\pi}^{\pi} \cos^{2n}\left(\frac{u-t}{2}\right) f(u)\, du .$$

3. For $a,b,c \in \mathbb{R}$ such that $c \geq b-a > 0$,
$$\beta(t) = \begin{cases} 1 - \left(\frac{t}{c}\right)^2 & , |t| \leq c \\ 0 & , |t| > c, \end{cases}$$
gives the Landau operators L_n on the space of all bounded, complex valued, measurable functions vanishing outside the interval (a,b) defined by
$$L_n(f;t) = \frac{1}{I(n)} \int_a^b \left[\left(\frac{c^2 - (u-t)^2}{c^2}\right)\right]^n f(u)\, du ,$$
where
$$I(n) = \int_{-c}^{c} \left(\frac{c^2-u^2}{c^2}\right)^n du.$$

Operators of the type $U_n(\cdot,t)$ were considered for the first time by Korovkin [31, chapter 1]. Later, Sikkema and Rathore [58] obtained results on the convergence in the basic approximation of continuous functions and under the assumption that $\beta''(0) \neq 0$, formulae of Voronvskaja type, giving a precise rate of this convergence for IInd differentiable functions, Lipschitz-Nikolskii constants and convergence in the simultaneous approximation of derivatives of functions with the corresponding derivatives of the operators. Next, Sikkema [59] constructed a subclass B' of B which yields a much higher order of approximation than any power of n. Also, he [69] obtained asymptotic formulae of Voronovskaya type for the speed with which $U_n(f;t) \to 0$ as $n \to \infty$.

Very recently, Winslin [67] has obtained global direct, inverse and saturation theorems for the linear combinations in L_p-norm $(1 \leq p < \infty)$. His inverse theorems correspond to the orders $n^{-\alpha/2}$ of approximation $(0 < \alpha < 2k+2)$.

Our aim here is to study the local approximation by the linear combinations $U_n(\cdot,k,t)$ of the convolution operators $U_n(\cdot,t)$ for $f \in L_p(\mathbb{R})$. It is clear that $U_n f$ is meaningful for $f \in L_p(\mathbb{R})$ and moreover $||U_n f||_{L_p(-\infty,\infty)} \leq ||f||_{L_p(-\infty,\infty)}$. Further, for $f \in L_p(-\infty,\infty)$, $||U_n f - f||_{L_p(\mathbb{R})} \to 0$ as $n \to \infty$ $(1 \leq p < \infty)$.

Let $d_0, \ldots, d_k$ be $k+1$ distinct positive numbers. Then for $n \geq \max_{0 \leq i \leq k} d_j^{-1}$, the linear combinations $U_n(f,k,t)$ are defined by

$$U_n(f,k,t) = \sum_{j=0}^{k} C(j,k)\, U_{d_j n}(f;t),$$

where

$$C(j,k) = \begin{cases} \prod\limits_{\substack{i=0 \\ i \neq j}}^{k} \dfrac{d_j}{d_j - d_i} & \text{if } k \neq 0 \\ 1 & \text{if } k = 0 . \end{cases}$$

In this chapter, for the operators $U_n(\cdot,k,t)$ with $\beta \in B^{\infty,2}$, we obtain local direct and inverse theorems of a more general order $\varphi(n^{-1/2})$ in L_p-norm $(1 \leq p < \infty)$ over contracting subintervals. All through this chapter, unless stated otherwise, the operators $U_n(\cdot,t)$ correspond to $\beta \in B^{\infty,2}$. It may be remarked here that our results corresponding to $k = 0$ constitute local approximation by the operators U_n themselves.

In the sequel, Section 1.2 introduces the general order φ of approximation and Section 1.3 contains basic results about the operators $U_n(\cdot,t)$ and $U_n(\cdot,k,t)$. The local direct theorem about $U_n(\cdot,k,t)$ is proved in Section 1.4. Lastly, in Sections 1.5 and 1.6, respectively, we obtain our main results, viz., $O(\varphi)$ and $o(\varphi)$-inverse theorems.

1.2 <u>The order φ of approximation.</u>

This section consists exclusively of the results about the order φ of approximation. The results here include the φ-generalisation of a lemma of Berens and Lorentz [12] .

<u>Definition</u> 1.2.1.

For any $r \in \mathbb{N}$, a function φ on $(0,c)$ is said to belong to the class Φ_r if it satisfies the following :

1. For any $0 < h < 1$, there exists a constant $\bar{K}_\varphi(h)$ depending only on φ and h such that, for all $t \in (0,c]$,

$$\frac{\varphi(t)}{\varphi(th)} \leq \bar{K}_\varphi(h)$$

2. For any $\delta > 0$,

$$K_\delta = \sup_{h \geq \delta} K_\varphi(h) < \infty$$

and lastly,

3. $h^r \bar{K}_\varphi(h) \to 0$ as $h \to 0$.

We give some illustrative examples of $\varphi \in \Phi_r$ at the end of this section.

Lemma 1.2.2. Let $c > 0$, Ω be an increasing function on $(0,c]$ and for some $r \in \mathbb{N}$, $\varphi \in \bar{\Phi}_r$ such that, for all $t,h \in (0,c]$,

(1.2.1) $$\Omega(h) \leq M\{\varphi(t) + (\frac{h}{t})^r \Omega(t)\} .$$

Then

(1.2.2) $$\Omega(t) = O(\varphi(t)) \ (t \to 0).$$

Remark 1. The aforementioned lemma of Berens and Lorentz [10] is a special case of this lemma obtained by taking $\varphi(t) = t^\alpha (0 < \alpha < r)$.

Remark 2. This lemma would be extensively used here and in subsequent chapters, in the proofs of $O(\varphi)$-inverse theorems.

Proof of Lemma 1.2.2. Since $h^r K_\varphi(h) \to 0$ as $h \to 0$, choose $A > 1$ such that $A^{-r} K_\varphi(A^{-1}) < \frac{1}{2M}$. Let $h_m = cA^{1-m}$ and

$M_1 = \max \{\frac{\Omega(c)}{\varphi(c)}, 2M K_\varphi(A^{-1})\}.$

First, we prove that

$$(1.2.3) \qquad \Omega(h_m) \leq M_1 \varphi(h_m) \quad \text{for } m = 1,2,\ldots$$

by induction on m.

Since $h_1 = c$, (1.2.3) is true for $m = 1$.

Assuming (1.2.3) for m-1 and taking $h = h_m$ and $t = h_{m-1}$ in (1.2.1),

$$\begin{aligned} \Omega(h_m) &\leq M \{\varphi(h_{m-1}) + A^{-r} \Omega(h_{m-1})\} \\ &\leq M\{K_\varphi(A^{-1}) + A^{-r} M_1 K_\varphi(A^{-1})\} \varphi(h_m) \\ &\leq M_1 \varphi(h_m). \end{aligned}$$

Thus (1.2.3) is proved for m and hence is true for $m = 1,2,\ldots$

Now, $t \in (0,c]$ implies that $h_m < t \leq h_{m-1}$ for some $m \in \mathbb{N}$. Hence,

$$\begin{aligned} \Omega(t) &\leq \Omega(h_{m-1}), \text{ which from (1.2.3) becomes} \\ &\leq M_1 \varphi(h_{m-1}) \\ &\leq M_1 K_\varphi(\frac{t}{h_{m-1}}) \varphi(t) \\ &\leq MK_{A^{-1}} \varphi(t), \end{aligned}$$

since $\frac{t}{h_{m-1}} \geq \frac{h_m}{h_{m-1}} = A^{-1}$ and hence the lemma is proved.

The next lemma would be of crucial importance in the proofs of $o(\varphi)$ inverse theorems.

Lemma 1.2.3. Let $c > 0$, $\varphi \in \Phi_r$ for some $r \in \mathbb{N}$ and $\Omega(t)$ be a positive increasing function on $(0,c]$ such that, for all $t,h \in (0,c]$,

$$(1.2.4) \qquad \Omega(t) \leq M\{\psi(t) + (\tfrac{h}{t})^r \Omega(t)\},$$

where

$$(1.2.5) \qquad \psi(t) = o(\varphi(t)) \quad (t \to 0).$$

Then

$$\Omega(t) = o(\varphi(t)) \quad (t \to 0).$$

Proof. From (1.2.5), $\psi(t) = e(t)\,\varphi(t)$, where

$$e(t) \to 0 \text{ as } t \to 0.$$

Since $\varphi \in \Phi_r$, $t^r K_\varphi(t) \to 0$ as $t \to 0$. Hence, we can choose $A > 1$ such that

$$A^{-r} K_\varphi(A^{-1}) < \frac{1}{4M}.$$

Let $h_m = cA^{1-m}$, $\delta_m = \dfrac{\Omega(h_m)}{\varphi(h_m)}$ and $\varepsilon > 0$.

Then, there exists $n_0 \in \mathbb{N}$ such that, for all $m \geq n_0$,

$$e(h_{m-1})\, K_\varphi(A^{-1})\, M < \varepsilon/4.$$

Taking $h = h_m$ and $t = h_{m-1}$ in (1.2.4) we get

$$\begin{aligned}
\Omega(h_m) &\leq M\{\psi(h_{m-1}) + A^{-r}\,\Omega(h_{m-1})\}\\
&= M\{e(h_{m-1})\,\varphi(h_{m-1}) + A^{-r}\,\delta_{m-1}\,\varphi(h_{m-1})\}\\
&\leq M\{e(h_{m-1})\,K_\varphi(A^{-1})\,\varphi(h_m) + A^{-r}\,K_\varphi(A^{-1})\,\delta_{m-1}\varphi(h_m)\}\\
&< (\varepsilon/4 + \frac{\delta_{m-1}}{4})\,\varphi(h_m)
\end{aligned}$$

and hence

$$\delta_m < \varepsilon/4 + \frac{\delta_{m-1}}{4}.$$

Thus, proceeding recursively,

$$\delta_{m+n_o} < \varepsilon/4 + \varepsilon/4^2 + \dots + \varepsilon/4^m + \frac{\delta_{n_o}}{4^m}$$

$$= (\varepsilon/3)(1 - \frac{1}{4^m}) + \frac{\delta_{n_o}}{4^m} .$$

Since ε is arbitrary, we get that $\delta_m \to 0$ as $m \to \infty$, which implies that

$$\Omega(h_m) = o(\varphi(h_m)) \quad (m \to \infty)$$

Now, $t \in (0,c]$ implies that there exists $m \in \mathbb{N}$ such that $h_m < t \leq h_{m-1}$. Thus

$$\Omega(t) \leq \Omega(h_{m-1}) = \delta_{m-1}\, \varphi(h_{m-1})$$

$$\leq \delta_{m-1}\, K_\varphi(t/h_{m-1})\, \varphi(t)$$

$$\leq \delta_{m-1}\, K_{A^{-1}}\, \varphi(t) \qquad (\because \;\; t/h_{m-1} \geq A^{-1}).$$

Therefore,

$$\Omega(t) = o(\varphi(t)) \qquad (t \to 0).$$

<u>Lemma</u> 1.2.4. Let $k \in \mathbb{N}$, $f \in L_p[a,b]$ $(1 \leq p < \infty)$ and $\varphi \in \Phi_k$ such that

(1.2.6) $\quad \omega_{k+1}(f,t,p,[a,b]) = O(\varphi(t)) \qquad (t \to 0).$

Then

$$\omega_k(f,t,p,[a,b]) = O(\varphi(t)) \qquad (t \to 0).$$

Remark : This lemma is a φ-generalisation of the following well-known result (see [65], § 3.34, §3.3.12-13, pp. 110-12] and [61]). In the sequel we put $\bar{\Phi}(t) = \Phi(c)$, $t \in (c,\infty)$.

Proof of the lemma 1.2.4 : By hypothesis (1.2.6), we have

$$\int_t^{\frac{b-a}{2k}} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}}\, du \le M_1 \int_t^{\frac{b-a}{2k}} \bar{\varphi}(u)\, u^{-k-1}\, du$$

and hence choosing $h (0 < h < 1)$ such that $h^r \bar{K}_{\varphi}(h) < 1$, we get

$$\int_t^{\frac{b-a}{2k}} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}}\, du \le M_1 \sum_{P=0}^{\infty} \int_{t/h^P}^{t/h^{P+1}} \varphi(u)\, u^{-k-1}\, du$$

$$\le M_1 \sum_{P=0}^{\infty} \varphi(t/h^P) \int_{t/h^P}^{t/h^{P+1}} \frac{\varphi(u)}{\varphi(t/h^P)}\, u^{-k-1}\, du$$

$$\le M_1' K_h \sum_{P=0}^{\infty} \bar{\varphi}(t/h^P) \int_{t/h^P}^{t/h^{P+1}} u^{-k-1}\, du,$$

since $t/uh^P \ge h$. Thus,

$$\int_t^{\frac{b-a}{2k}} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}}\, du \le \frac{M_2}{t^k} \sum_{P=0}^{\infty} \varphi\left(\frac{t}{h^P}\right) h^{Pk}.$$

But,

$$\frac{\varphi(t/h^P)}{\varphi(t)} = \frac{\varphi(t/h^P)}{\varphi(t/h^{P-1})} \frac{\varphi(t/h^{P-1})}{\varphi(t/h^{P-2})} \cdots \frac{\varphi(t/h)}{\varphi(t)} \le (\bar{K}_{\varphi}(h))^P .$$

Hence, using the fact that $K_{\varphi}(h)\, h^k < 1$,

$$\int_t^{\frac{b-a}{2k}} \omega_{k+1}(f,u,p,[a,b])\, du \le \frac{M_3}{t^k} \bar{\varphi}(t) .$$

Using a result of [65] , viz., for any $f \in L_p[a,b]$ $(1 \leq p < \infty)$,

(1.2.7) $$\omega_k(f,t,p,[a,b]) \leq M_k t^k \{\|f\|_{L_p[a,b]} + \int_t^{\frac{b-a}{2k}} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}} du\},$$

we have

(1.2.8) $$\omega_k(f,t,p,[a,b]) \leq M_k t^k \{\|f\|_{L_p[a,b]} + \frac{M_3}{t^k} \varphi(t)\} .$$

Now, since $\varphi \in \Phi_k$, $h^k K_\varphi(h) \to 0$ as $h \to 0$. Thus, we get that

$$t^k/\varphi(t) \to 0 \text{ as } t \to 0.$$

Hence, (1.2.8) implies that

$$\omega_k(f,t,p,[a,b]) = O(\varphi(t)) \quad (t \to 0).$$

The little -o version of lemma 1.2.4 is

<u>Lemma</u> 1.2.5 : Let $k \in \mathbb{N}$, $f \in L_p[a,b]$ $(1 \leq p < \infty)$ and $\varphi \in \Phi_k$ such that

$$\omega_{k+1}(f,t,p,[a,b]) = o(\varphi(t)) \quad (t \to 0) .$$

Then

$$\omega_k(f,t,p,[a,b]) = o(\varphi(t)) \quad (t \to 0).$$

<u>Proof</u> : Let $\varepsilon > 0$. Since $\omega_{k+1}(f,t,p,[a,b]) = o(\varphi(t))$, we can choose a sufficiently small $\delta > 0$ such that, for all $t \in (0,\delta]$,

$$\omega_{k+1}(f,t,p,[a,b]) \leq \varepsilon \varphi(t).$$

Hence,

$$\int_t^\delta \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}} du \leq \varepsilon \int_t^\infty \frac{\varphi(u)}{u^{k+1}} du .$$

Thus, proceeding as in the proof of lemma1.2.4, we get

$$(1.2.9)\qquad \int_t^{\delta} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}}\,du \le \frac{\varepsilon M_3}{t^k}\,\bar{\varphi}(t).$$

Next,

$$(1.2.10)\qquad \int_{\delta}^{\frac{b-a}{2k}} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}}\,du \le \omega_{k+1}\left(f,\frac{b-a}{2k},p,[a,b]\right)\frac{\delta^{-k}}{k}$$

$$= M_4, \text{ say.}$$

Now, from (1.2.9) and (1.2.10),

$$\int_t^{\frac{b-a}{2k}} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}}\,du \le \frac{\varepsilon M_3}{t^k}\,\bar{\varphi}(t) + M_4 .$$

Thus, from (1.2.7),

$$\omega_k(f,t,p,[a,b]) \le M_k t^k \|f\|_{L_p[a,b]}$$

$$+ M_k t^k \int_t^{\frac{b-a}{2k}} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}}\,du$$

$$\le M_k\, t^k\, \|f\|_{L_p[a,b]} + M_3'\,\varepsilon\,\bar{\varphi}(t) + M_4\, M_k\, t^k$$

$$= M_k'\, t^k\, \|f\|_{L_p[a,b]} + M_k''\,\varepsilon\,\bar{\varphi}(t), \text{ say.}$$

Hence,

$$\frac{\omega_k(f,t,p,[a,b])}{\bar{\varphi}(t)} \le M_k' \frac{t^k}{\bar{\varphi}(t)} \|f\|_{L_p[a,b]} + M_k''\,\varepsilon .$$

Thus, using the fact that

$t^k/\bar{\varphi}(t) \to 0$ as $t \to 0$, we get

$$\limsup_{t\to 0} \frac{\omega_k(f,t,p,[a,b])}{\bar{\varphi}(t)} \le \varepsilon\, M_k'' .$$

Since ε is arbitrary, we get the result.

Examples of approximation orders $\varphi \in \Phi_r$:

Throughout our discussion in the rest of this section, we take $r \in \mathbb{N}$, $\alpha \in \mathbb{R}$ such that $0 < \alpha < r$.

1. For $t \in (0,c]$, $c > 0$,

$$\varphi(t) = t^{\alpha} \in \Phi_r.$$

For, here we can take $\bar{K}_{\varphi}(h) = \frac{1}{h^{\alpha}}$. Then clearly the other conditions are also satisfied.

2. Let $p \in \mathbb{N}$, $q > 0$ and $c > 0$ be such that $\log^p 1/c > 0$. Then, for $t \in (0,c]$,

$$\varphi(t) = t^{\alpha}(\log^p 1/t)^q \quad \in \Phi_r.$$

For, here

$$\sup_t \frac{\varphi(t)}{\varphi(th)} \leq 1/h^{\alpha}$$

and hence, taking $K_{\varphi}(h) = 1/h^{\alpha}$, we observe that the other two conditions of Φ_r are also satisfied.

3. Let $p \in \mathbb{N}$, $q > 0$ and $c > 0$ be such that $\log^p 1/c > 0$. Then, for

$$\varphi(t) = t^{\alpha}/(\log^p 1/t)^q \quad , \ t \in (0,c],$$

$$\bar{K}_{\varphi}(h) = \sup_t \frac{\varphi(t)}{\varphi(th)} = \frac{1}{h^{\alpha}} \left(\frac{\log^p 1/ch}{\log^p 1/c}\right)^q$$

and hence $\varphi \in \Phi_r$.

4. Let $-\infty \leq a < b \leq \infty$, $k \in \mathbb{N}$. Let $f \in L_p[a,b]$ $(1 \leq p \leq \infty)$ be a function which is not a polynomial of degree $\leq k-1$. Then, for r $(> k)$ $\in \mathbb{N}$,

$$\varphi(t) = \omega_k(f,t,p,[a,b]) \in \Phi_r$$

where $t \in (0,c]$, $c < \frac{b-a}{2k}$. For, by (Timan [65]),

$$\frac{\bar{\varphi}(t)}{\bar{\varphi}(th)} \leq (1+\frac{1}{h})^k = \bar{K}_\varphi(h)$$

and $\bar{K}_\varphi(h)$ satisfies the required conditions.

1.3. Basic Results

In this section we give some basic estimates and results about the operators $U_n(\cdot,t)$ and $U_n(\cdot,k,t)$, which will be used in the later sections.

Lemma 1.3.1 [58] : Let $k \in \mathbb{N}^0$ and $\beta \in B$. Then

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{k} (-1)^i \binom{k}{i} \alpha(n+i+1)}{\sum_{i=0}^{k} (-1)^i \binom{k}{i} \alpha(n+i)} = 1 ,$$

where $\alpha(n)$ is given by (1.1.2).

Taking $k = 0$ in lemma 1.3.1 and applying it repeatedly we obtain

Corollary 1.3.2 [58] : For $m \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{\alpha(n-m)}{\alpha(n)} = 1 ,$$

where $\alpha(n)$ is given by (1.1.2).

The following lemma gives an asymptotic expansion of the moments of the operators $U_n(\cdot,t)$.

Lemma 1.3.3 Let $\delta > 0$, $\beta \in B_\delta$ and $k \in \mathbb{N}$. Then the following asymptotic expansion is valid:

$$\frac{1}{\alpha(n)} \int_{-\infty}^{\infty} t^k \beta^n(t)\,dt \sim \sum_{r=\left[\frac{k+1}{2}\right]}^{\infty} \frac{C_{k,r}}{n^r}, \quad (n \to \infty)$$

where $C_{k,r}$ are constants independent of n.

Proof: Assuming that $\beta \in B$ is m-times continuously differentiable in a neighbourhood of the origin with $\beta''(0) \neq 0$ and $t^k \beta^\ell(t) \in L_1(-\infty,\infty)$ for some $\ell \geq 1$, Winslin [67] has shown that

$$\frac{1}{\alpha(n)} \int_{-\infty}^{\infty} t^k \beta^n(t)\,dt = \sum_{r \geq \left[\frac{k+1}{2}\right]} \frac{C_{k,r}}{n^r} + o\left(\frac{1}{n^{\frac{k+m-2}{2}}}\right),$$

where $C_{k,r}$ do not depend on n.

Since $\beta \in B_\delta$, these assumptions are satisfied for an arbitrary m and hence the lemma.

Corollary 1.3.4 Let $k \in \mathbb{N}$, $\delta > 0$ and $\beta \in B_\delta$. Then

$$U_n(|u-t|^k; t) \leq \frac{M}{n^{k/2}}, \quad M \text{ being a constant.}$$

Proof Let $s > k$ be an even integer. Then by Hölders inequality

$$\frac{1}{\alpha(n)} \int_{-\infty}^{\infty} \beta^n(t-u)\, |u-t|^k\, du$$

$$\leq \left\{\frac{1}{\alpha(n)} \int_{-\infty}^{\infty} \beta^n(t-u)\,|u-t|^s du\right\}^{k/s} \times$$

$$\times \left\{\frac{1}{\alpha(n)} \int_{-\infty}^{\infty} \beta^n(t-u)\,du\right\}^{1-k/s}$$

Now the lemma follows from lemma 1.3.3.

The following lemma has a crucial importance all over this chapter. It is a tool with which we can extend our results for any $\beta \in B^{\infty,2}$ by just proving them for $\beta \in B_\delta$ for an appropriate $\delta > 0$.

<u>Lemma</u> 1.3.5 Let $\beta, \beta^* \in B$ such that $\beta = \beta^*$ on $(-\delta, \delta)$ for some $\delta > 0$. Then for any $\ell > 0$, and $f \in L_p(-\infty,\infty)$ $(1 \leq p < \infty)$,

$$||U_n(f,t) - U^*_n(f;t)||_{L_p(-\infty,\infty)} = O(n^{-\ell})\ ||f||_{L_p}(-\infty,\infty)\quad (n\to\infty)$$

where U_n, U_n^* are the operators corresponding to β and β^* respectively.

<u>Proof</u>
$$\int_{-\infty}^{\infty} \beta^n(u)\ f(t-u)\,du = \int_{-\delta}^{\delta} \beta^n(u)\, f(t-u)\,du + \int_{|u|\geq\delta} \beta^n(u)\, f(t-u)\,du$$
$$= f_n(t) + g_n^\beta(t), \text{ say.}$$

Then, assuming that $\chi_\delta(u)$ is the characteristic function of $\mathbb{R} - (-\delta,\delta)$, we have

$$(1.3.1)\qquad ||g_n^\beta(t)||_{L_p(-\infty,\infty)} \leq \int_{-\infty}^{\infty} \chi_\delta(u)\, \beta^n(u)\, ||f||_{L_p(-\infty,\infty)}\,du$$
$$\leq \alpha_\beta(1)\ \beta_\delta^{n-1}\ ||f||_{L_p(-\infty,\infty)},$$

where

$$\beta_\delta = \sup_{\mathbb{R}-(-\delta,\delta)} \beta(t) < 1 \qquad \text{and}$$

$$\alpha_\beta(1) = \int_{-\infty}^{\infty} \beta(u)\,du.$$

Write

$$\alpha(n) = \alpha_\beta(n) = \int_{-\delta}^{\delta} \beta^n(u)\,du + \int_{|u|\geq\delta} \beta^n(u)\,du$$
$$= \gamma_n + \delta_n^\beta, \qquad \text{say.}$$

Clearly

$$(1.3.2) \qquad \delta_n^{\beta} \leq \alpha_{\beta}(1)\ \beta_{\delta}^{n-1} .$$

Let $\delta^* \leq \delta$ be such that, for $|u| < \delta^*$,

$$(1.3.3) \qquad \beta(u) > \max\left\{\frac{1+\beta_\delta}{2}, \frac{1+\beta_\delta^*}{2}\right\} = \lambda , \quad \text{say.}$$

Then

$$(1.3.4) \qquad \lambda > \max\{\beta_\delta^*, \beta_\delta\} \quad \text{and}$$

$$(1.3.5) \qquad \gamma_n \geq 2\delta\lambda^n > 2\delta^*\lambda^n .$$

Hence

$$||U_n f - U_n^* f||_{L_p(-\infty,\infty)} = \left|\left|\frac{f_n + g_n^{\beta}}{\gamma_n + \delta_n^{\beta}} - \frac{f_n + g_n^{\beta*}}{\gamma_n + \delta_n^{\beta*}}\right|\right|_{L_p(-\infty,\infty)}$$

$$\leq \frac{1}{\alpha_\beta(n)\cdot\alpha_{\beta*}(n)}\{||f_n||_{L_p(-\infty,\infty)}(\delta_n^{\beta} + \delta_n^{\beta*})$$

$$+ \alpha_{\beta*}(n)||g_n^{\beta}||_{L_p(-\infty,\infty)} + \alpha_\beta(n)||g_n^{\beta*}||_{L_p(-\infty,\infty)}$$

Thus from (1.3.1), (1.3.2), (1.3.5) and using the fact that $||f_n||_{L_p(-\infty,\infty)} \leq ||f||_{L_p(-\infty,\infty)}$, we get that

$$||U_n f - U_n^* f||_{L_p(-\infty,\infty)} \leq ||f||_{L_p(-\infty,\infty)}\left\{\frac{\alpha_\beta(1)\,\beta_\delta^{n-1} + \alpha_{\beta*}(1)\,\beta_\delta^{*\,n-1}}{2\delta^*\lambda^n}\right\}$$

$$+ \frac{||f||_{L_p(-\infty,\infty)}}{2\delta^*\lambda^n}\,\alpha_\beta(1)\,\beta_\delta^{n-1} + \frac{||f||_{L_p(-\infty,\infty)}\,\alpha_{\beta*}(1)\,\beta_\delta^{*n-1}}{2\delta^*\lambda^n}$$

Now, since $\frac{\beta_\delta}{\lambda} < 1$, for any $\ell > 0$,

$$\frac{\alpha_\beta(1)\,\beta_\delta^{n-1}}{2\delta^*\lambda^n} = \frac{\alpha_\beta(1)}{2\delta^*\lambda}\cdot\left(\frac{\beta_\delta}{\lambda}\right)^{n-1} = O(n^{-\ell})$$

Similarly, for any $\ell > 0$,

$$\frac{\alpha_{\beta}(1)\,\beta_{\delta}^{*n-1}}{2\delta^{*}\lambda^{n}} = O(n^{-\ell}) \quad \text{and hence}$$

$$||U_n t - U_n^* f||_{L_p(-\infty,\infty)} = ||f||_{L_p(-\infty,\infty)} \cdot O(n^{-\ell})$$

Hence the result.

Now we prove two localisation lemmas which would be found useful in the later sections.

Throughout the rest of this chapter, we use notation $I_j = [a_j, b_j]$, $j = 1,2$ where $-\infty<a_1<a_2<b_2<b_1<\infty$.

Lemma 1.3.6 Let $f \in L_p(-\infty,\infty)$ $(1 \le p < \infty)$ and $\beta \in B_\delta$ with $\delta < \min(a_2-a_1, b_1-b_2)$. Then, for any $\ell > 0$,

$$||U_n(f;t)||_{L_p(I_2)} \le ||f||_{L_p(I_1)} \cdot$$

Proof By Jensen's inequality

$$||U_n(f;t)||^p_{L_p(I_2)} \le \frac{1}{\alpha(n)} \int_{a_2}^{b_2} \int_{-\infty}^{\infty} \beta^n(u)\,|f(t-u)|^p \,du\,dt$$

$$= \frac{1}{\alpha(n)} \int_{a_2}^{b_2} \int_{-\delta}^{\delta} \beta^n(u)\,|f(t-u)|^p \,du\,dt$$

$$\le \frac{1}{\alpha(n)} \int_{a_2-\delta}^{b_2+\delta} \int_{-\delta}^{\delta} \beta^n(u)\,|f(t)|^p \,du\,dt$$

$$\le ||f||^p_{L_p(I_1)}$$

and hence the lemma.

The following lemma is clear from the definition of the operator $U_n(\cdot,t)$.

Lemma 1.3.7 Let $a < c < d < b$ and $\chi(u)$ be the characteristic function of the interval $[a,b]$, further, let $f \in L_p(-\infty,\infty)$ $(1 \leq p < \infty)$ and $\beta \in B_\delta$ with $\delta < \min(c-a, b-d)$. Then, for all $t \in [c,d]$

$$U_n((1-\chi(u))f(u);t) = 0$$

Next we prove a lemma which gives an asymptotic expansion of the error for functions in $C_0^{2k+2}(-\infty,\infty)$.

Lemma 1.3.8 Let $f \in C_0^{2k+2}(-\infty,\infty)$ and $\beta \in B_\delta$ for any $\delta > 0$. Then there holds

$$U_n(f,k,t)-f(t) = n^{-(k+1)}\{\sum_{i=1}^{2k+2} c_i f^{(i)}(t)+o(n^{-(k+1)})\ (n \to \infty),$$

uniformly for $t \in \mathbb{R}$, where c_i's are certain constants.

Proof For some ξ lying between u and t we have

$$(1.3.6) \qquad f(u) = \sum_{i=0}^{2k+2} \frac{f^{(i)}(t)}{i\,!}(u-t)^i + \\ + \frac{(u-t)^{2k+2}}{(2k+2)!}(f^{(2k+2)}(\xi)-f^{(2k+2)}(t)).$$

From lemma 1.3.3 and the fact that $\sum_{j=0}^{k} C(j,k)\, d_j^{-m}=0$, $m=1,2,\ldots,k$, it follows that for some constants c_i,

$$(1.3.7) \qquad U_n((u-t)^i,\ k,t) = n^{-(k+1)}c_i + o(n^{-(k+1)}) \quad (n \to \infty),$$

where $i=1,2,\ldots,2k+2$ and o-term is uniformly in $t \in (-\infty,\infty)$.

Since $f^{(2k+2)} \in C_0(-\infty,\infty)$, given an arbitrary $\varepsilon > 0$, $\exists$ $\delta > 0$ such that whenever $|x-y| < \delta$,

$$|f^{(2k+2)}(x) - f^{(2k+2)}(y)| < \varepsilon.$$

Hence,

$$\begin{aligned}
&|U_n((u-t)^{2k+2}(f^{(2k+2)}(\xi)-f^{(2k+2)}(t));t)| \\
&\leq \varepsilon|U_n(|u-t|^{2k+2},t)| + \frac{2}{\delta^2}\,||f^{(2k+2)}||_{C_o(-\infty,\infty)} \times \\
&\qquad\qquad \times |U_n(|u-t|^{2k+4};t)| \\
&\leq \varepsilon.\ \frac{M}{n^{k+1}} + \frac{M_1}{n^{k+2}}
\end{aligned}$$

Thus

$$(1.3.8)\quad |U_n((u-t)^{2k+2}(f^{(2k+2)}(\xi) - f^{(2k+2)}(t));t)| \leq \frac{M_2}{n^{k+1}}(\varepsilon + \frac{1}{n}).$$

Now, combining (1.3.7) and (1.3.8) from (1.3.6) we get the result.

We close this section by stating a special case of a lemma of Sikkema and Rathore [58] .

<u>Lemma</u> 1.3.9 Let $\beta \in B_\delta$. Then for any $m < n$, there holds the identity:

$$\frac{\partial^m}{\partial u^m}\beta^n(u-t) = \beta^{n-m}(u-t)\sum_{k=0}^{m}\sum_{\nu=0}^{[\frac{m-k}{2}]} n^{\nu}(n\frac{\partial}{\partial u}\beta(u-t))^k g_{\nu,k,m}(u-t)$$

for all $t,u \in \mathbb{R}$, where $g_{\nu,k,m}$ are certain linear combinations of products of $\beta_1,\ldots,\beta^{(m)}$ (which are independent of n) and hence bounded on $\mathbb{R}$.

1.4 <u>Direct theorem</u>

In this section we prove a direct theorem whose corresponding inverse theorem is dealt with in the next section.

<u>Theorem</u> 1.4.1 Let $f \in L_p(-\infty,\infty)$, $\beta \in B^{\infty,2}$ and $\varphi \in \Phi_{2k+2}$. Then, for sufficiently large values of n

$$\omega_{2k+2}(f,t,p,I_1) = O(\varphi(t)) \quad (t \to o)$$

implies that

$$||U_n(f,k,t) - f(t)||_{L_p(I_2)} = O(\varphi(n^{-\frac{1}{2}})) \quad (n \to \infty).$$

Before proving the theorem, we prove some auxiliary results which shall be used in the proof of the theorem.

<u>Lemma</u> 1.4.2 Let $\beta \in B_\delta$ with $\delta < \min(b_1-b_2, a_2-a_1)$ and $f \in L_p(-\infty,\infty)$ $(1 \leq p < \infty)$ have 2k+2 derivatives on I_1 with $f^{(2k+1)} \in A.C.(I_1)$ and $f^{(2k+2)} \in L_p(I_1)$. Then

$$||U_n(f,k,t) - f(t)||_{L_p(I_2)} \leq \frac{M}{n^{k+1}} \{||f^{(2k+2)}||_{L_p(I_1)} + ||f||_{L_p(I_1)}\}$$

<u>Proof</u> Assuming that $X(u)$ is the characteristic function of I_1, we have from lemma 1.3.7, for all $t \in I_2$,

$$U_n(f;t) = \frac{1}{\alpha(n)} \int_{-\infty}^{\infty} \beta^n(t-u) X(u) f(u)\, du.$$

For $t \in I_2$ and $u \in I_1$, using

$$f(u) = \sum_{i=0}^{2k+1} \frac{(u-t)^i}{i!} f^{(i)}(t) + \frac{1}{(2k+1)!} \int_t^u (u-w)^{2k+1} f^{(2k+2)}(w)\, dw,$$

we get,

$$U_n(f;t) = \frac{1}{\alpha(n)} \sum_{i=0}^{2k+1} \frac{f^{(i)}(t)}{i!} \int_{-\infty}^{\infty} \beta^n(t-u) X(u) (u-t)^i\, du$$

$$+ \frac{1}{\alpha(n)} \frac{1}{(2k+1)!} \int_{-\infty}^{\infty} \beta^n(t-u) X(u) \int_t^u (u-w)^{2k+1} f^{(2k+2)}(w)\, dw\, du.$$

Thus

$$U_n(f,k,t)-f(t)=\sum_{i=1}^{2k+1}\frac{f^{(i)}(t)}{i!}\{\sum_{j=0}^{k}C(j,k)\frac{1}{\alpha(d_jn)}\int_{-\infty}^{\infty}\beta^{d_jn}(t-u)X(u)\times (u-t)^i du$$

$$+\frac{1}{(2k+1)!}\{\sum_{j=0}^{k}C(j,k)\{\frac{1}{\alpha(d_jn)}\int_{-\infty}^{\infty}\beta^{d_jn}(t-u)X(u)\times \int_t^u (u-w)^{2k+1} f^{(2k+2)}(w)\,dwdu\}\}$$

(1.4.1) $\qquad = Z_1+Z_2,$ say.

Since $\sum_{j=0}^{k}C(j,k)d_j^{-m}=0,\ m=1,2,\ldots,k$, using lemmas 1.3.3 we get

(1.4.2) $\qquad ||Z_1||_{L_p(I_2)}\le\frac{M}{n^{k+1}}\{||f^{(2k+2)}||_{L_p(I_2)}+||f||_{L_p(I_2)}\}.$

Consider

$$R=\int_{a_2}^{b_2}|\frac{1}{\alpha(n)}\int_{-\infty}^{\infty}X(u)\beta^n(t-u)\int_t^u(u-w)^{2k+1}f^{(2k+2)}(w)\,dw|^p dt$$

which by Jensen's inequality is

$$\le\frac{1}{\alpha(n)}\int_{a_2}^{b_2}\int_{a_1}^{b_1}\beta^n(t-u)|u-t|^{(2k+1)p}|\int_t^u X(u)f^{(2k+2)}(w)\,dw|^p dudt$$

$$\le\frac{1}{\alpha(n)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\beta^n(t)|t|^{(2k+1)p}|\int_{t+u}^{u}X(u)f^{(2k+2)}(w)\,dw|^p dudt.$$

Again using Jensen's inequality

$$R\le\frac{1}{\alpha(n)}\int_{-\infty}^{\infty}\beta^n(t)|t|^{(2k+1)p}\int_{-\infty}^{\infty}\int_{t+u}^{u}|t|^{p-1}|X(w)f^{(2k+2)}|^p dwdudt$$

$$=\frac{1}{\alpha(n)}\int_{-\infty}^{\infty}\beta^n(t)|t|^{(2k+1)p}\int_{-\infty}^{\infty}\int_{w-t}^{w}|t|^{p-1}|X(w)f^{(2k+2)}(w)|^p\,dudwdt$$

$$\leq \frac{1}{\alpha(n)} \int_{-\infty}^{\infty} \beta^n(t) \, |t|^{(2k+1)p} \, |t|^p \, ||f^{(2k+2)} \quad X \, ||^p_{L_p(-\infty,\infty)} \, dt$$

Thus,

$$(1.4.3) \qquad R \leq \frac{M}{n^{(k+1)p}} \, ||f^{(2k+2)}||^p_{L_p(I_1)}$$

Hence from (1.4.1) and (1.4.3), we get

$$(1.4.4) \qquad ||Z_2||_{L_p(I_2)} \leq \frac{M_1}{n^{k+1}} \, ||f^{(2k+2)}||_{L_p(I_1)} \; .$$

Combining (1.4.2) and (1.4.4), we get the result.

<u>Proof of theorem</u> 1.4.1. In view of lemma 1.3.5, it suffices to prove the theorem for $\beta \in B_\delta$ with $\delta < \min\,(a_2-x, y-b_2)$ where $a_1 < x < a_2 < b_2 < y < b_1$.

Now, for sufficiently small $\eta > 0$

$$||U_n(f,k,t) - f(t)||_{L_p(I_2)}$$

$$\leq ||U_n(f-f_{\eta,2k+2},k,t)||_{L_p(I_2)}$$

$$+ ||U_n(f_{\eta,2k+2},k,t)-f_{\eta,2k+2}(t)||_{L_p(I_2)}$$

$$+ ||f_{\eta,2k+2}(t) - f(t)||_{L_p(I_2)}$$

$$(1.4.5) \qquad = Z_1 + Z_2 + Z_3, \text{ say.}$$

Let X be the characteristic function of $[x,y]$. Then, from lemmas 1.3.7 and 1.3.6,

$$Z_1 \leq ||U_n(X(f-f_{\eta,2k+2}),k,t)||_{L_p(I_2)}$$

$$\leq ||f-f_{\eta,2k+2}||_{L_p[x,y]}$$

and hence by lemma 0.6.5

(1.4.6) $\quad Z_1 \leq M_1\{\omega_{2k+2}\ (f,\eta,p,I_1)\}.$

From lemma 1.4.2, we get

$$Z_2 \leq \frac{M_2}{n^{k+1}}\{||f^{(2k+2)}_{\eta,2k+2}||_{L_p[x,y]} + ||f_{\eta,2k+2}||_{L_p[x,y]}\},$$

which by lemma 0.6.5 implies

(1.4.7) $\quad Z_2 \leq \frac{M_3}{n^{k+1}}\{\eta^{-(2k+2)}\ \omega_{2k+2}(f,\eta,p,I_1) + ||f||_{L_p(I_1)}\}.$

Also, by the same lemma

(1.4.8) $\quad Z_3 \leq M_4\ \omega_{2k+2}\ (f,\eta,p,[x,y]).$

Choosing η such that $\eta = n^{-1/2}$, from (1.4.6-8), we get

(1.4.9) $\quad ||U_n(f,k,t) - f(t)||_{L_p(I_2)}$

$$\leq M_6\{\omega_{2k+2}(f,n^{-1/2},p,I_1)+n^{-(k+1)}||f||_{L_p(I_1)}\}.$$

Since $\varphi \in \Phi_{2k+2}$, $h^{2k+2}\ K_\varphi(h) \to 0$ as $h \to 0$ which implies that

$$t^{2k+2}/\varphi(t) \to 0 \text{ as } t \to C$$

and hence, from the hypothesis and (1.4.9), we get

$$||U_n(f,k,t) - f(t)||_{L_p(I_2)} = O(\varphi(n^{-1/2})) \qquad (n \to \infty)$$

which proves the theorem.

Corollary 1.4.3. Let $f \in L_p(-\infty,\infty)$ $(1 \le p < \infty)$, $\beta \in B^{\infty,2}$ and $\varphi \in \Phi_{2k+2}$. Then,

$$\omega_{2k+2}(f,t,p,I_1) = o(\varphi(t)) \qquad (t \to 0)$$

implies that

$$||U_n(f,k,t) - f(t)||_{L_p(I_2)} = o(\varphi(n^{-1/2})) \quad (n \to \infty).$$

Proof. Proceeding as in the proof of theorem 1.4.1, we get

$$(1.4.10) \qquad ||U_n(f,k,t) - f(t)||_{L_p(I_2)}$$

$$\le M\{\omega_{2k+2}(f,n^{-1/2},p,I_1)+n^{-(k+1)}||f||_{L_p(I_1)}\}.$$

From (1.4.10) and the fact that $t^{2k+2}/\varphi(t) \to 0$ as $t \to 0$, we get that the left hand side of (1.4.10) is of $o(\varphi(n^{-1/2}))$ $(n \to \infty)$ and hence the corollary follows.

1.5 O(φ)-inverse theorem

In theorem 1.4.1, we proved that if $\omega_{2k+2}(f,t,p,I_1) = O(\varphi(t))$ $(t \to 0)$ then

$$||U_n(f,k,t) - f(t)||_{L_p(I_2)} = O(\varphi(n^{-1/2})) \quad (n \to \infty).$$

Now, we prove the corresponding local inverse theorem.

Theorem 1.5.1. Let $\beta \in B^{\infty,2}$, $\varphi \in \Phi_{2k+2}$ and $f \in L_p(-\infty,\infty)$ $(1\le p<\infty)$. Then,

$$(1.5.1) \qquad ||U_n(f,k,t)-f(t)||_{L_p(I_1)} = O(\varphi(n^{-1/2})) \quad (n \to \infty)$$

implies that

$$(1.5.2)\quad \omega_{2k+2}(f,t,p,I_2) = O(\varphi(t)) \quad (t \to 0).$$

Before proceeding to the proof of the theorem, we prove the following :

<u>Lemma</u> 1.5.2. Let $f \in L_p(-\infty,\infty)$ $(1 \le p < \infty)$ and $\beta \in B_\delta$ with $\delta < \min\,(a_2-a_1, b_1-b_2)$. Then, for any $i,j \in \mathbb{N}^o$

$$\left\|U_n\left((u-t)^i \int_t^u (u-w)^j\, f(w)\, dw;\, t\right)\right\|_{L_p[a_2,b_2]} \le M\, n^{-\left(\frac{i+j+1}{2}\right)} \|f\|_{L_p[a_1,b_1]}.$$

<u>Proof</u> : In view of lemma 1.3.7, for all $t \in I_2$,

$$R = U_n\left((u-t)^i \int_t^u (u-w)^j\, f(w)\, dw;t\right)$$

$$= U_n\left(\chi(u)\,(u-t)^i \int_t^u (u-w)^j\, f(w)\, dw;t\right)$$

where $\chi(u)$ is the characteristic function of I_1.

Then, for all $t \in I_2$,

$$|R| \le \left| U_n\left(\chi(u)\ |u-t|^{i+j} \left|\int_t^u |f(w)|dw\right|;t\right)\right|.$$

Now, proceeding as in the case of the estimate of R in lemma 1.4.2, we get that

$$\|R\|_{L_p(I_2)} \le \frac{M}{n^{\frac{i+j+1}{2}}} \|f\|_{L_p(I_1)}$$

and hence the result.

<u>Lemma</u> 1.5.3 : Let $\beta \in B_\delta$, for some $\delta > 0$ and $m \in \mathbb{N}$. Then, for $f \in L_p(-\infty,\infty)$ $(1 \le p < \infty)$ such that supp $f \subset (a_2,b_2)$,

$$||U_n^{(2m)}(f;t)||_{L_p(I_2)} = O(n^m ||f||_{L_p(I_2)}) \qquad (n \to \infty).$$

Further, if also, $f^{(2m-1)}$ exists and is absolutely continuous on $[a_2,b_2]$ and $f^{(2m)} \in L_p[a_2,b_2]$ then for all n sufficiently large

$$||U_n^{(2m)}(f;t)||_{L_p(I_2)} \le M ||f^{(2m)}||_{L_p(I_2)},$$

M being a constant.

<u>Proof</u> :For $n > 2m$, it is clear that

$$U_n^{(2m)}(f;t) = \frac{1}{\alpha(n)} \int_{-\infty}^{\infty} \left(\frac{\partial^{2m}}{\partial t^{2m}} \beta^n(t-u)\right) f(u)\, du.$$

Thus, by lemma 1.3.9, we get

$$U_n^{(2m)}(f;t) = \frac{1}{\alpha(n)} \sum_{k=0}^{2m} \sum_{\nu=0}^{[\frac{2m-k}{2}]} n^{\nu+k} \int_{-\infty}^{\infty} \beta^{n-2m}(t-u)\left(\frac{\partial}{\partial t}\beta(t-u)\right)^k \times$$
$$\times f(u)\, g_{\nu,k,m}(t-u)\, du$$

where $g_{\nu,k,m}$ are certain linear combinations of products of $\beta, \beta', \ldots, \beta^{(2m)}$ which are independent of n and hence bounded on $\mathbb{R}$.

Now, since $\beta'(0) = 0$, for some ξ lying between 0 and $t-u$, we have $\beta'(t-u) = (t-u)\,\beta''(\xi)$. Using this, we get

$$U_n^{(2m)}(f;t) = \sum_{k=0}^{2m} \sum_{\nu=0}^{[\frac{2m-k}{2}]} n^{\nu+k} \frac{1}{\alpha(n)} \int_{-\infty}^{\infty} (t-u)^k \beta^{n-2m}(t-u)(\beta''(\xi))^k \times$$
$$\times f(u)\, g_{\nu,k,m}(t-u)\, du$$

Using the boundedness of β'' and $g_{\nu,k,m}$ on $\mathbb{R}$,

$$|U_n^{(2m)}(f;t)| \leq M \sum_{k=0}^{2m} \sum_{\nu=0}^{\left[\frac{2m-k}{2}\right]} n^{\nu+k} \frac{1}{\alpha(n)} \int_{-\infty}^{\infty} \beta^{n-2m}(t-u)\,|t-u|^k \times$$
$$\times |f(u)|\; du,$$

and hence, from corollary 1.3.2, for $f \in L_1(-\infty,\infty)$,

$$\int_{-\infty}^{\infty} |U_n^{(2m)}(f;t)|\; dt$$
$$\leq M_1 \sum_{k=0}^{2m} \sum_{\nu=0}^{\left[\frac{2m-k}{2}\right]} n^{\nu+k} \frac{1}{\alpha(n-2m)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \beta^{n-2m}(u)\; |u-t|^k |f(t-u)|\,du dt$$
$$= M_1 \sum_{k=0}^{2m} \sum_{\nu=0}^{\left[\frac{2m-k}{2}\right]} \frac{n^{\nu+k}}{\alpha(n-2m)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \beta^{n-2m}(u)\,|u-t|^{k}\; |f(t-u)|\; du\; dt$$
$$\leq M_2 \sum_{k=0}^{2m} \sum_{\nu=0}^{\left[\frac{2m-k}{2}\right]} n^{\nu+k}\; n^{-k/2}\; ||f||_{L_1(-\infty,\infty)}$$
$$\leq M_3\; n^m\; ||f||_{L_1(-\infty,\infty)}$$

and hence

$$||U_n^{(2m)}(f;t)||_{L_1(-\infty,\infty)} \leq M_3\; n^m\; ||f||_{L_1(-\infty,\infty)}\,.$$

Similarly, we can prove that

$$||U_n^{(2m)}(f;t)||_{L_\infty(-\infty,\infty)} \leq M_4\; n^m\; ||f||_{L_\infty(-\infty,\infty)}\,.$$

Hence, by (Riesz-Thorin's interpolation) lemma 0.6.4 , for $1 \leq p \leq \infty$

$$||U_n^{(2m)}\; f||_{L_p(-\infty,\infty)} \leq M_5\; n^m\; ||f||_{L_p(-\infty,\infty)}\,,$$

from which the first part of the lemma follows.

Since $f^{(m-1)}$ is absolutely continuous, $f^{(m)} \varepsilon L_p(-\infty,\infty)$ and $\beta \varepsilon B_\delta$, we can write

$$U_n^{(2m)}(f;t) = U_n(f^{(2m)};t),$$

from which the second part of the lemma is clear.

Proof of the theorem 1.5.1 : Let (x_i, y_i), $i = 1,2,3,4,5$ satisfy $x_o = a_1 < x_1 < x_2 < x_3 < x_4 < x_5 < a_2 < b_2 < y_5 < y_4 < y_3 < y_2 < y_1 < b_1 = y_o$. Choose $g \varepsilon C_o^{2k+2}$ such that supp $g \subset (x_4, y_4)$ and $g(t) = 1$ for $t \varepsilon [x_5, y_5]$. Let $\delta < \min (x_3-x_2, y_2-y_3)$. Then, in view of lemma 1.3.5, it suffices to prove the theorem for $\beta \varepsilon B_\delta$.

Writing $fg = \bar{f}$, we have, by applying Jensen's inequality 2k+2 times

$$\left|\int_o^\gamma \cdots \int_o^\gamma U_n^{(2k+2)}(\bar{f},k,t+\sum_{i=1}^{2k+2} z_i)\, dz_1 \ldots dz_{2k+2}\right|^p$$

$$\leq \gamma^{(2k+2)(p-1)} \{\int_o^\gamma \cdots \int_o^\gamma |U_n^{(2k+2)}(\bar{f},k,t+\sum_{i=1}^{2k+2} z_i)|^p \, dz_1 \ldots dz_{2k+2} .$$

Hence, by Fubini's theorem, for all γ sufficiently small $(\gamma < (2k+2)^{-1} \min \{x_3-x_2, y_2-y_3\})$

$$\int_{x_3}^{y_3} \left|\int_o^\gamma \cdots \int_o^\gamma U_n^{(2k+2)}(\bar{f},k,t+\sum_{i=1}^{2k+2} z_i)\, dz_1 \ldots dz_{2k+2}\right|^p dt$$

$$\leq \gamma^{(2k+2)(p-1)} \int_o^\gamma \cdots \int_o^\gamma \int_{x_3}^{y_3} | U_n^{(2k+2)}(\bar{f},k,t+\sum_{i=1}^{2k+2} z_i)|^p \, dt\, dz_1 \ldots dz_{2k+2}$$

$$\leq \gamma^{(2k+2)p} \, ||U_n^{(2k+2)}(\bar{f},k,t)||^p_{L_p[x_2,y_2]}.$$

Thus, from lemma 0.6.1

$$||\Delta_\gamma^{2k+2} \, U_n(\bar{f},k,t)||_{L_p[x_3,y_3]} \leq \gamma^{2k+2} \, ||U_n^{(2k+2)}(\bar{f},k,t)||_{L_p[x_2,y_2]}$$

$$\leq \gamma^{2k+2} \, \{||U_n^{(2k+2)}(\bar{f}-\bar{f}_{\eta,2k+2},k,t)||_{L_p[x_2,y_2]}$$

$$+ ||U_n^{(2k+2)}(\bar{f}_{\eta,2k+2},k,t)||_{L_p[x_2,y_2]}.$$

Let η be sufficiently small ($\eta < \{4(k+1)\}^{-2} \times$

$$\times \min_{0\leq i\leq 4} \{x_{i+1}-x_i, y_i-y_{i+1}\}).$$

Then, using lemmas 1.5.3 and 0.6.5, it is easily seen that

$$(1.5.3) \quad ||\Delta_\gamma^{2k+2} \, U_n(\bar{f},k,t)||_{L_p[x_3,y_3]}$$

$$= M_1 \, \gamma^{2k+2}(n^{k+1} + \frac{1}{\eta^{2k+2}}) \, \omega_{2k+2}(\bar{f},\eta,p,[x_3,y_3]) \, .$$

Next, we show that

$$(1.5.4) \qquad ||\Delta_\gamma^{2k+2} \, \{U_n(\bar{f},k,t) - \bar{f}(t)\}||_{L_p[x_3,y_3]} = O(\varphi(n^{-\frac{1}{2}}))\,(n \to \infty).$$

After having proved (1.5.4), combining with (1.5.3), we get

$$||\Delta_\gamma^{2k+2} \, \bar{f}(t)||_{L_p[x_3,y_3]} \leq M_2\{\varphi(n^{-1/2}) + \gamma^{2k+2}(n^{k+1} + \frac{1}{\eta^{2k+2}}) \times$$

$$\times \, \omega_{2k+2}(\bar{f},\eta,p,\, [x_3,y_3])\} .$$

Choosing n such that $n \leq \eta^{-2} < 2n$, and taking supremum over $0 < \gamma \leq t$, for all t sufficiently small, we have

$$\omega_{2k+2}(\bar{f},t,p,[x_3,y_3])$$

$$\leq M_3 \{\varphi(\eta) + (\frac{t}{\eta})^{2k+2} \omega_{2k+2}(\bar{f},\eta,p,[x_3,y_3])\}$$

which by lemma 1.2.2 implies that

$$\omega_{2k+2}(\bar{f},t,p,[x_3,y_3]) = O(\varphi(t)) \quad (t \to 0).$$

The assertion (1.5.2) follows from the fact that $\bar{f}(t) = f(t)$ for $t \in [a_2,b_2]$.

To prove (1.5.4) it suffices to prove that

(1.5.5) $$||U_n(\bar{f},k,t) - \bar{f}(t)||_{L_p[x_3,y_3]} = O(\varphi(n^{-1/2})) \quad (n \to \infty).$$

We prove (1.5.5) by an induction as follows :

First, we prove the assertion (1.5.5), for all $\varphi \in \Phi_1$. Next, assuming the assertion (1.5.5) for all $\varphi \in \Phi_r$, $1 \leq r \leq 2k+1$, we prove it for $\varphi \in \Phi_{r+1}$.

Let $\varphi \in \Phi_1$. Then

$$||U_n(fg,k,t) - (fg)(t)||_{L_p[x_3,y_3]}$$

$$\leq ||U_n((f(u)-f(t))g(t),k,t)||_{L_p[x_3,y_3]}$$

$$+ ||U_n(f(u)(g(u)-g(t)),k,t)||_{L_p[x_3,y_3]}$$

$$= ||g(t)\{U_n(f,k,t) - f(t)\}||_{L_p[x_3,y_3]} +$$

$$+ \| U_n(f(u)(u-t)\ g'(\xi), k, t) \|_{L_p[x_3, y_3]}$$

for some ξ lying between u and t

$$= T_1 + T_2, \text{ say.}$$

From hypothesis (1.5.1),

(1.5.6) $\quad T_1 \le M_4\ \varphi(n^{-1/2})$.

To estimate T_2, consider

$$R = \| U_n((u-t)\ f(u)\ g'(\xi); t) \|^p_{L_p[x_3, y_3]}$$

$$= \int_{x_3}^{y_3} \left| \frac{1}{\alpha(n)} \int_{-\infty}^{\infty} \beta^n(t-u)(u-t)\ f(u)\ g'(\xi)\ du \right|^p dt$$

$$\le \frac{1}{\alpha(n)} \int_{x_3}^{y_3} \int_{-\infty}^{\infty} \beta^n(t-u)\ |u-t|^p\ |f(u)|^p\ |g'(\xi)|^p\ du\ dt,$$

by Jensen's inequality. Now, using Fubini's theorem and the boundedness of g',

$$R \le \frac{M_5}{\alpha(n)} \int_{-\infty}^{\infty} \int_{x_3}^{y_3} \beta^n(t-u)\ |u-t|^p\ |f(u)|^p\ dt\ du$$

$$= \frac{M_5}{\alpha(n)} \int_{-\infty}^{\infty} \beta^n(u)\ |u|^p \int_{x_3}^{y_3} |f(t-u)|^p\ dt\ du$$

$$= \frac{M_5}{\alpha(n)} \int_{-\delta}^{\delta} \beta^n(u)\ |u|^p \int_{x_3}^{y_3} |f(t-u)|^p\ dt\ du$$

$$\le \frac{M_5}{\alpha(n)} \int_{-\delta}^{\delta} \beta^n(u)\ |u|^p \int_{a_1}^{b_1} |f(t)|^p\ dt\ du.$$

Hence, from corollary 1.3.4,

$$R \leq \frac{M_6}{n^{p/2}} ||f||^p_{L_p[a_1,b_1]}$$

which implies that

(1.5.7) $$T_2 \leq \frac{M_7}{n^{1/2}} ||f||_{L_p[a_1,b_1]}.$$

Now, $\varphi \varepsilon \Phi_1$ implies that $hK_\varphi(h) \to 0$ as $h \to 0$, which in turn implies that

(1.5.8) $$t/\varphi(t) = o(1), \quad (t \to 0).$$

Combining (1.5.6-8), we get the assertion (1.5.5) for all $\varphi \varepsilon \Phi_1$. Next, assume the assertion (1.5.5) for all $\varphi \varepsilon \Phi_r$, $1 \leq r \leq 2k+1$. Let $\varphi \varepsilon \Phi_{r+1}$. Then

$$||U_n(fg,k,t) - (fg)(t)||$$

$$\leq ||U_n((f(u)-f(t))g(t),k,t)||_{L_p[x_3,y_3]}$$

$$+ ||U_n(f(u)(g(u)-g(t)),k,t)||_{L_p[x_3,y_3]}$$

$$\leq M_7\ \varphi(n^{-1/2}) + ||U_n((f(u)-f_{\eta,2k+2}(u)) \times$$

$$\times (g(u)-g(t)),k,t)||_{L_p[x_3,y_3]}$$

$$+ ||U_n((f_{\eta,2k+2}(u)-f_{\eta,2k+2}(t))(g(u)-g(t)),k,t)||_{L_p[x_3,y_3]}$$

$$+ ||U_n(f_{\eta,2k+2}(t)(g(u)-g(t)),k,t)||_{L_p[x_3,y_3]}$$

$$= M_8\ \varphi(n^{-1/2}) + Z_1 + Z_2 + Z_3, \text{ say.}$$

By lemmas 1.3.8 and lemma 0.6.5

$$(1.5.9) \qquad Z_3 \leq \frac{M_9}{n^{k+1}} ||f||_{L_p[x_2,y_2]}.$$

For some ξ lying between u and t

$$Z_1 = ||U_n((f(u)-f_{\eta,2k+2}(u))(u-t)g'(\xi),k,t)||_{L_p[x_3,y_3]}.$$

Proceeding as in the case of T_2 above, we get

$$(1.5.10) \qquad Z_1 \leq M_{10} \{n^{-1/2} ||f-f_{\eta,2k+2}||_{L_p[x_2,y_2]}\}.$$

For some ξ lying between u and t,

$$Z_2 \leq \frac{1}{(2k+1)!} ||U_n(\sum_{i=1}^{2k} \frac{g^{(i)}(t)}{i!} (u-t)^i \times$$

$$\times (\int_t^u (u-w)^{2k+1} f_{\eta,2k+2}^{(2k+2)}(w)\, dw),k,t)||_{L_p[x_3,y_3]}$$

$$+ \frac{1}{((2k+1)!)^2} ||U_n(g^{(2k+1)}(\xi)(u-t)^{2k+1} \times$$

$$\times (\int_t^u (u-w)^{2k+1} f_{\eta,2k+2}^{(2k+2)}(w)\, dw),k,t)||_{L_p[x_3,y_3]}$$

$$+ \sum_{i=1}^{2k+1} \sum_{j=1}^{2k} \frac{1}{i!j!} ||f_{\eta,2k+2}^{(i)}(t)\, g^{(j)}(t) \times$$

$$\times U_n((u-t)^{i+j},k,t)||_{L_p[x_3,y_3]}$$

$$+ \frac{1}{(2k+1)!} \sum_{i=1}^{2k+1} \frac{1}{i!} ||f_{\eta,2k+2}^{(i)}(t) \times$$

$$\times U_n((u-t)^{2k+1+i} g^{(2k+1)}(\xi),k,t)||_{L_p[x_3,y_3]}$$

$$= R_1 + R_2 + R_3 + R_4, \text{ say.}$$

From lemma 1.5.2, we get

(1.5.11) $$R_1 \leq M_{11} \{n^{-(k+3/2)} \|f^{(2k+2)}_{\eta,2k+2}\|_{L_p[x_2,y_2]}\}$$

and

(1.5.12) $$R_2 \leq M_{12} \{n^{-(2k+3/2)} \|f^{(2k+2)}_{\eta,2k+2}\|_{L_p[x_2,y_2]}\}.$$

Since $\sum_{j=0}^{k} C(j,k) d_j^{-m} = 0$, $m = 1,2,\dots,k$, from lemmas 1.3.3 and 0.6.3

(1.5.13) $$R_3 \leq \frac{M_{13}}{n^{k+1}} \{\|f^{(2k+1)}_{\eta,2k+2}\|_{L_p[x_3,y_3]} + \|f_{\eta,2k+2}\|_{L_p[x_3,y_3]}\}$$

and

(1.5.14) $$R_4 \leq \frac{M_{14}}{n^{k+1}} \{\|f^{(2k+1)}_{\eta,2k+2}\|_{L_p[x_3,y_3]} + \|f_{\eta,2k+2}\|_{L_p[x_3,y_3]}\}$$

Thus, from (1.5.9-14) choosing n such that $n \leq \eta^{-2} < 2n$, it follows from lemma 0.6.5 , that

(1.5.15) $$Z_1+Z_2+Z_3 \leq M_{15}\{n^{-1/2}\, \omega_{2k+2}(f,n^{-1/2},p,[x_1,y_1]) + n^{-1/2}\, \omega_{2k+1}(f,n^{-1/2},p,[x_1,y_1]) + n^{-(k+1)} \|f\|_{L_p[x_1,y_1]}\}$$

Let $\varphi^*(t) = \frac{\varphi(t)}{t}$. Then

$$\frac{\varphi^*(t)}{\varphi^*(th)} = \frac{\varphi(t)}{t}\ \frac{th}{\varphi(th)} = h\, \frac{\varphi(t)}{\varphi(th)}$$

and hence we may take $K_{\varphi^*}(h) = h\, K_\varphi(h)$,

which, since $\varphi \in \Phi_{r+1}$, implies that $\varphi^* \in \Phi_r$.

Hence, by induction hypothesis

$$\|U_n(\bar{f},k,t) - \bar{f}(t)\|_{L_p[x_3,y_3]} = O(\varphi^*(n^{-1/2})) \qquad (n \to \infty).$$

This, in turn, implies (1.5.2), with φ replaced by φ^* and hence, since I_2 is arbitrary subject only to $I_2 \subset (a_1,b_1) = I_1^o$, we can conclude that, for the interval $[x_1,y_1]$,

$$(1.5.16) \qquad \omega_{2k+2}(f,t,p,[x_1,y_1]) = O(\varphi^*(t)) \qquad (t \to 0).$$

Also, since $r \leq 2k+1$, $\varphi^* \in \Phi_{2k+1}$.

Thus, from (1.5.16) and lemma 1.2.4, we get

$$(1.5.17) \qquad \omega_{2k+1}(f,t,p,[x_1,y_1]) = O(\varphi^*(t)) \qquad (t \to 0).$$

Now, from (1.5.15-17)

$$(1.5.18) \qquad Z_1+Z_2+Z_3 \leq M_{16}\{n^{-1/2}\varphi^*(n^{-1/2}) + n^{-1/2}\varphi^*(n^{-1/2}) + n^{-(k+1)}\|f\|_{L_p[x_1,y_1]}\}.$$

Since $\varphi \in \Phi_{r+1} \subset \Phi_{2k+2}$,

$$(1.5.19) \qquad t^{2k+2}/\varphi(t) \to 0 \text{ as } t \to 0.$$

Hence, from (1.5.18-19)

$$Z_1+Z_2+Z_3 \leq M_{17}\,\varphi(n^{-1/2})$$

and hence the theorem follows.

<u>Corollary</u> 1.5.4 : Let $1 \leq m \leq 2k+2$, $\varphi \in \Phi_m$, $\beta \in B^{\infty,2}$ and $f \in L_p(-\infty,\infty)$ $(1 \leq p < \infty)$. Then

$$||U_n(f,k,t) - f(t)||_{L_p(I_1)} = O(\varphi(n^{-1/2})) \qquad (n \to \infty)$$

implies that

$$\omega_m(f,t,p,I_2) = O(\varphi(t)) \qquad (t \to 0).$$

<u>Proof</u> : Since $1 \leq m \leq 2k+2$ and $\varphi \in \Phi_m$, $\varphi \in \Phi_{2k+2}$. Hence, by theorem 1.5.1,

$$\omega_{2k+2}(f,t,p,I_2) = O(\varphi(t)) \qquad (t \to 0).$$

Again, $\varphi \in \Phi_m$ implies that

$$h^m K_\varphi(h) \to 0 \qquad (h \to 0)$$

which implies, for $j = m, m+1, \ldots, 2k+1$, that

$$(1.5.20) \qquad h^j K_\varphi(h) \to 0 \qquad (h \to 0).$$

Now, lemma 1.2.2 and (1.5.20) for $j = 2k+1$, imply that

$$\omega_{2k+1}(f,t,p,I_2) = O(\varphi(t)) \qquad (t \to 0)$$

which by (1.5.20), for $j = 2k$, gives

$$\omega_{2k}(f,t,p,I_2) = O(\varphi(t)) \qquad (t \to 0).$$

Continuing in this manner, for $j = m$, (1.5.20) implies that

$$\omega_m(f,t,p,I_2) = O(\varphi(t)) \qquad (t \to 0)$$

which completes the proof.

1.6 $o(\varphi)$-inverse theorem

In corollary 1.4.3, we obtained that if

$$\omega_{2k+2}(f,t,p,I_1) = o(\varphi(t)) \qquad (t \to 0)$$

then

$$||U_n(f,k,t)-f(t)||_{L_p(I_2)} = o(\varphi(n^{-1/2})) \qquad (n \to \infty).$$

Now, we prove the corresponding local $o(\varphi)$-inverse theorem.

Theorem 1.6.1. Let $\beta \in B^{\infty,2}$, $\varphi \in \Phi_{2k+2}$ and $f \in L_p(-\infty,\infty)$ $(1\le p<\infty)$. Then

$$||U_n(f,k,t) - f(t)||_{L_p(I_1)} = o(\varphi(n^{-1/2})) \qquad (n \to \infty)$$

implies that

$$(1.6.1) \qquad \omega_{2k+2}(f,t,p,I_2) = o(\varphi(t)) \qquad (t \to 0).$$

Proof : Let (x_i,y_i), $i = 1,2,3,4,5$ be such that $x_o = a_1 < x_1 < x_2 < x_3 < x_4 < x_5 < a_2 < b_2 < y_5 < y_4 < y_3 < y_2 < y_1 < b_1 = y_o$. In view of the lemma 1.3.5, we can assume that $\beta \in B_\delta$ with $\delta < \min(x_3-x_2, y_2-y_3)$. Choose $g \in C_o^{2k+2}$ such that $\operatorname{supp} g \subset (x_4,y_4)$ and $g(t) = 1$ for $t \in [x_5,y_5]$. Writing $fg = \bar{f}$ and proceeding as in the proof of the theorem 1.5.1, for sufficiently small $\eta,\gamma > 0$, we get

$$(1.6.2) \qquad ||\Delta_\gamma^{2k+2}\, U_n(\bar{f},k,t)||_{L_p[x_3,y_3]}$$

$$\le M_1\, \gamma^{2k+2}\left(n^{k+1} + \frac{1}{\eta^{2k+2}}\right)\omega_{2k+2}(\bar{f},\eta,p,[x_3,y_3]).$$

Next, we shall prove that

$$(1.6.3) \qquad \|\Delta_\gamma^{2k+2}\{\bar{f}(t)-U_n(\bar{f},k,t)\}\|_{L_p[x_3,y_3]} = o(\varphi(n^{-1/2}))\ (n\to\infty).$$

After having proved (1.6.3), we define $\psi(t)$ as follows :

$$\psi(t)=\begin{cases}\|\Delta_\gamma^{2k+2}\{\bar{f}(t)-U_n(\bar{f},k,t)\}\|_{L_p[x_3,y_3]} & \text{if } t=n^{-1/2} \text{ for any } n\in\mathbb{N}\\ \psi((n^{-1/2}) & \text{if } t\in((n+1))^{-1/2},\ n^{-1/2}) \text{ for any } n\in\mathbb{N}.\end{cases}$$

Clearly

$$\psi(t) = o(\varphi(t)) \qquad (t\to 0).$$

Combining (1.6.2) and (1.6.3), we get

$$\|\Delta_\gamma^{2k+2}\ \bar{f}(t)\|_{L_p[x_3,y_3]} \leq M_2\{\psi(n^{-1/2})$$

$$+\ \gamma^{2k+2}(n^{k+1} + \frac{1}{\eta^{2k+2}})\ \omega_{2k+2}(\bar{f},\eta,p,[x_3,y_3]\,)\}\ .$$

Choosing n such that $n \leq \eta^{-2} < n+1$, for all sufficiently small t, it implies that

$$\omega_{2k+2}\ (\bar{f},t,p,[x_3,y_3]$$

$$\leq M_3\ \{\psi(\eta) + (\frac{t}{\eta})^{2k+2}\ \omega_{2k+2}(\bar{f},\eta,p,[x_2,y_2])\},$$

which, by lemma 1.2.3, gives

$$\omega_{2k+2}(\bar{f},t,p,[x_3,y_3]) = o(\varphi(t)) \qquad (t\to 0).$$

The conclusion (1.6.1), now, follows since $\bar{f}(t) = f(t)$ for $t\in I_2$.

We prove (1.6.3) by induction as in the theorem 1.5.1. That is, first we prove that the assertion (1.6.3) is true for all $\varphi\in\Phi_1$. Assuming that the assertion (1.6.3) is true for

(1.6.5) $\omega_{2k+2}(\bar{f},t,p,[x_1,y_1]) = o(\varphi^*(t)) \quad (t \to \infty)$.

Since $\varphi^* \varepsilon \Phi_{2k+1}$, by lemma 1.2.5, we get

(1.6.6) $\omega_{2k+1}(\bar{f},t,p,[x_1,y_1]) = o(\varphi^*(t)) \quad (t \to 0)$.

Now, $\varphi \varepsilon \Phi_{r+1} \subset \Phi_{2k+2}$, implies that

$$h^{2k+2} K_{\varphi}(h) \to 0 \quad \text{as } h \to 0$$

which implies that

(1.6.7) $t^{2k+2}/\varphi(t) \to 0$ as $t \to 0$.

Combining (1.6.4-7), we get

$$Z_1+Z_2+Z_3 = o(\varphi(n^{-1/2}))$$

which proves (1.6.3) and hence the theorem.

<u>Corollary</u> 1.6.2 : Let $\beta \varepsilon B^{\infty,2}$, $\varphi \varepsilon \Phi_m$, $1 \leq m \leq 2k+2$ and $f \varepsilon L_p(-\infty,\infty)$ $(1 \leq p < \infty)$. Then

$$||U_n(f,k,t) - f(t)||_{L_p(I_1)} = o(\varphi(n^{-1/2})) \quad (n \to \infty)$$

implies that

$$\omega_m(f,t,p,I_2) = o(\varphi(t)) \quad (t \to 0).$$

<u>Proof</u> : Since $\varphi \varepsilon \Phi_m \subset \Phi_{2k+2}$, theorem 1.6.1 implies that

$$\omega_{2k+2}(f,t,p,I_2) = o(\varphi(t)) \quad (t \to 0).$$

Again, $\varphi \in \Phi_m$, implies that

$$h^m K_\varphi(h) \to 0 \qquad (h \to 0)$$

which, in turn, implies that, for $j = m, m+1, \ldots, 2k+1,$

$$(1.6.8) \qquad h^j K_\varphi(h) \to 0 \qquad (h \to 0).$$

Taking $j = 2k+1$, in (1.6.8), from lemma 1.2.5, we get

$$\omega_{2k+1}(f,t,p,I_2) = o(\varphi(t)) \qquad (t \to 0)$$

which, again by (1.6.8) for $j = 2k$, gives

$$\omega_{2k}(f,t,p,I_2) = o(\varphi(t)) \qquad (t \to 0).$$

Continuing in this manner, for $j = m$, (1.6.8) implies that

$$\omega_m(f,t,p,I_2) = o(\varphi(t)) \qquad (t \to 0).$$

CHAPTER II

φ-INVERSE THEOREMS FOR LINEAR COMBINATIONS AND INTERPOLATORY MODIFICATIONS OF BERNSTEIN-KANTOROVITCH POLYNOMIALS

2.1 Introduction

In this chapter we study the linear combinations $P_n(\cdot,k,t)$ and interpolatory modifications $P_{n,m}(\cdot,t)$ of the Bernstein-Kantorovitch polynomials P_n which themselves are a modification of Bernstein polynomials suggested by Kantorovitch [29] for approximation of functions belonging to $L_p[0,1]$, given by

$$P_n(f,t) = (n+1)\left\{\sum_{\nu=0}^{n} p_{n\nu}(t) \int_{\frac{\nu}{(n+1)}}^{\frac{(\nu+1)}{(n+1)}} f(u)\,du\right\}$$

where

$$(2.1.1) \qquad p_{n\nu}(t) = \binom{n}{\nu} t^{\nu}(1-t)^{n-\nu}.$$

For any $k+1$ distinct positive integers $d_0,\ldots,d_k$, the linear combinations $P_n(\cdot,k,t)$ of $P_n(\cdot,t)$ are defined by

$$P_n(\cdot,k,t) = \sum_{j=0}^{k} C(j,k)\, P_{d_j n}(\cdot,t), \quad n \in \mathbb{N}$$

where

$$C(j,k) = \begin{cases} \displaystyle\prod_{\substack{i=0\\ i\neq j}}^{k} \frac{d_j}{d_j-d_i} & \text{if } k \neq 0 \\ 1 & \text{if } k = 0 \end{cases}$$

The interpolatory modifications, $P_{n,m}(\cdot,t)$, introduced by Sinha [61], of the Bernstein-Kantorovitch polynomials $P_n(\cdot,t)$, are modifications of $P_n(\cdot,t)$ by making use of the classical Newton's interpolation polynomials. This is accomplished by replacing function value $f(u)$ at the points 'u' by Newton's interpolation polynomial of mth degree based at the nodes $u, u+\frac{1}{n^{1/2}}, \ldots, u+\frac{m}{n^{1/2}}$.

Thus, for any $f \in L_p[0,1]$ $(1 \leq p < \infty)$, interpolatory modifications $P_{n,m}(f;t)$ of order m, of the Bernstein-Kantorovitch polynomials $P_n(f;t)$ are defined as follows:

$$P_{n,m}(f;t) = (n+1)\sum_{\nu=0}^{n}\sum_{j=0}^{m}\frac{n^{j/2}}{j!}\,p_{n\nu}(t)\int_{\frac{\nu}{n+1}}^{\frac{\nu+1}{n+1}}\left(\prod_{i=0}^{j-1}\left(t-u-\frac{i}{n^{1/2}}\right)\right)\times \Delta^{j} f(u)\,du,$$

where $\prod_{i=0}^{j-1}\left(t-u-\frac{i}{n^{1/2}}\right)$ for $j=0$ is interpreted as 1 and $f(u)$ is regarded as zero when $u > 1$. Here and in the sequel Δ denotes $\Delta_{n^{-1/2}}$.

Approximation by Bernstein-Kantorovitch polynomials in L_p-norms $(1 \leq p < \infty)$ has been studied by Lorentz [34], Butzer [15], Hoeffding [26], Bojanic and Shisha [13], Ditzian and May [22], Grundmann [24], Muller [44], Maier [37,38], Riemenschneider [54], May [40], Becker and Nessel [7,8] and recently by Winslin [67]. These works establish that the optimal rate of convergence for the Bernstein-Kantorovitch polynomials in L_p-norms is $O(n^{-1})$,

which is also the optimal rate of convergence for the corresponding original operators, viz., the Bernstein polynomials with respect to sup-norm.

Ditzian and May [22] obtained local direct, inverse and saturation theorems in L_p-norms in the set up of contracting subintervals. Grundmann [24] and Muller [44] obtained bounds for the error in L_p-approximation for the cases $p=1$ and $p>1$, respectively, in terms of the first order integral modulus of smoothness of the functions. Similar estimates were obtained in weighted L_1-norm by Bojanic and Shisha [13] . Bounds for the error in the approximation of differentiable functions have been obtained by Muller [44] and Hoeffding [26] respectively.

Maier [37] proved the global saturation theorem for $P_n(\cdot,t)$ in $L_1[0,1]$-norm, which has been extended to $L_p[0,1](1 < p < \infty)$ by Riemenschineider [54] . Becker and Nessel [7] characterised the saturation class of Bernstein-Kantorovitch polynomials and May [40] proved a modified global saturation theorem alongwith a correction term in a weighted $L_p[0,1]$-norm $(1 \leq p < \infty)$.

Sinha [61] obtained $O(n^{-\alpha/2})$ local direct, inverse and saturation theorems for linear combinations and interpolatory modifications of Bernstein-Kantorovitch polynomials in L_p-norm $(1 \leq p < \infty)$ over contracting subintervals. Recently, Winslin [67] has obtained

global direct, inverse theorems for linear combinations of Bernstein-Kanotorovitch polynomials, in L_p-norm $(1 \leq p < \infty)$, for the order $O(n^{-\alpha/2})$.

In the present chapter, we extend the results of Sinha [61] to a more general order $O(\varphi(n^{-1/2}))$ for linear combinations and interpolatory modifications of Bernstein-Kantorovitch polynomials in L_p-norm $(1 \leq p < \infty)$. These results are also local in nature over contracting subintervals. In this chapter, section 2.2 contains some basic results about the operators $P_n(\cdot,t)$, $P_{n,m}(\cdot,t)$ and $P_n(\cdot,k,t)$ which shall be used in the later sections. In section 2.3 we obtain $O(\varphi)$-inverse theorem for $P_n(\cdot,k,t)$ and section 2.4 consists of $o(\varphi)$-inverse theorem for $P_n(\cdot,k,t)$. Similarly, sections 2.5 and 2.6, respectively contain $O(\varphi)$- and $o(\varphi)$-inverse theorems for the operator $P_{n,m}(\cdot,t)$.

2.2 Basic Results

Throughout the rest of this chapter $I = [0,1]$, $I_j = [a_j,b_j]$, $j=1,2,3$ where $0 < a_j < a_{j+1}$ and $b_{j+1} < b_j < 1$.

Lemma 2.2.1 [35] : Let $s \in \mathbb{N}$. Then $T_{n,s}(t)$ defined as

$$T_{n,s}(t) = \sum_{\nu=0}^{n} (\nu-nt)^s \, p_{n\nu}(t)$$

is a polynomial in t and n; in t of degree $\leq s$, in n of degree $[s/2]$, where $[\cdot]$ stands for the integral part and $p_{n\nu}(t)$ is given by (2.1.1). $T_{n,2s}(t)$ depends only

on $t(1-t)$ and n with the coefficient of n^s being $\frac{(2s)!}{2^s s!}(t(1-t))^s$. $T_{n,2s+1}(t)$ is a polynomial in $t(1-t)$ and n multiplied by the factor $(1-2t)$.

Lemma 2.2.2: Let r be a positive number. Then, for $t \in [0,1]$,

$$|T_{n,r}(t)| \leq Mn^{r/2}; \quad M \text{ being a constant independent}$$

of t and n.

May [40] expressed the moments of Bernstein-Kantorovitch polynomials in terms of the moments of the Bernstein polynomials as follows:

Lemma 2.2.3: Let $m \in \mathbb{N}$ and $p(t) = t(1-t)$. Then

$$P_n((u-t)^m ;t) = \frac{n+1}{(m+1)\ p(t)} B_{n+1} ((u-t)^{m+2};t)$$

where

$$B_n (f;t) = \sum_{\nu=0}^{n} p_{n\nu}(t) f(\nu/n)$$

and

$p_{n\nu}(t)$ is given by (2.1.1).

From lemmas 2.2.1 and 2.2.3, we get

Corollary 2.2.4: For $m \in \mathbb{N}$, there holds

$$P_n((u-t)^m ;t) = \frac{1}{(n+1)^{m+1}} Q(n+1;t)$$

where $Q(n+1;t)$ is a polynomial in $(n+1)$ of degree $[\frac{m+2}{2}]$ and in t of degree $\leq m$.

Corollary 2.2.5: Let $r \in \mathbb{N}$. Then, for $t \in I$,

$$P_n(|u-t|^r ; t) \le \frac{M}{n^{r/2}},$$

M being a constant independent of t and n.

Lemma 2.2.6 ([22]): Let $f \in L_p[0,1]$ $(1 \le p < \infty)$. Then

$$\|P_n(f;t)\|_{L_p[0,1]} \le \|f\|_{L_p[0,1]}.$$

Sinha [61] introduced the duals P_n^* of the operators P_n as follows:

Definition 2.2.7: The dual operator sequence $\{P_n^*\}$ of $\{P_n\}$ is defined as

$$P_n^*(f;u) = \int_0^1 K(n,t,u)\, f(t)\,dt$$

where $K(n,t,u)$ is given by

$$K(n,t,u) = (n+1)\{\sum_{\nu=0}^{n} p_{n\nu}(t)\, \chi_{n\nu}(u)\},$$

(2.2.1) $\left\{\begin{array}{l} \chi_{n\nu}(u) \text{ is the characteristic function of} \\ \text{the interval } \left[\frac{\nu}{n+1}, \frac{\nu+1}{n+1}\right) \text{ for } \nu = 0,1,\ldots,n-1 \\ \text{and of } \left[\frac{n}{n+1}, 1\right] \text{ for } \nu = n. \end{array}\right.$

The following lemma of Sinha [61] gives the order of the moments of the dual operator P_n^*.

Lemma 2.2.8: Let $k \in \mathbb{N}$ and $u \in I_1$. Then

$$P_n^*(|u-t|^k;u) = O(n^{-k/2}) \quad (n \to \infty)$$

uniformly in $u \in I_1$.

Ditzian and May[22] proved the following

Lemma 2.2.9: Let $f \in L_p[0,1]$ $(1 \le p < \infty)$, $i \in \mathbb{N}^0$, $[a_1, b_1] \subset (a,b)$ and $\chi(u)$ denote the characteristic function of $[a,b]$. Then, for any fixed positive number ℓ,

$$\|P_n(f(u)(u-t)^i(1-\chi(u));t)\|_{L_p[a_1,b_1]} = O(n^{-\ell})\ \|f\|_{L_p[0,1]}.$$

The proof of the above result in [22] seems somewhat ambiguous. An alternative proof of the same may be given as follows:

With $K(n,t,u)$ as in (2.2.1), $\ell > 0$ and $0 < \delta < \min(a_1-a,\ b-b_1)$, using Jensen's inequality, we have

$$\begin{aligned}
&\|P_n(f(u)(u-t)^i(1-\chi(u));\ t)\|^p_{L_p[a_1,b_1]} \\
&= \int_{a_1}^{b_1} \left|\int_0^1 K(n,t,u)\ f(u)(u-t)^i(1-\chi(u))\,du\right|^p dt \\
&\le \int_{a_1}^{b_1} \int_0^1 K(n,t,u)\ |f(u)|^p |u-t|^{ip}(1-\chi(u))\,du\ dt \\
&\le \int_{a_1}^{b_1} \int_0^1 K(n,t,u)\ |f(u)|^p \frac{|u-t|^{ip+\ell}}{\delta^\ell}(1-\chi(u))\,du\ dt \\
&\le \int_0^1 |f(u)|^p \left(\int_0^1 K(n,t,u)\ \frac{|u-t|^{ip+\ell}}{\delta^\ell}\,dt\right) du \\
&\le \frac{M}{n^{(ip+\ell)/2}}\ \|f\|^p_{L_p[0,1]},
\end{aligned}$$

by corollary 2.2.5. Since ℓ is arbitrary, we get the result.

The moment estimates of the operators $P_{n,m}(\cdot,t)$ were obtained by Sinha [61] and are given in

Lemma 2.2.10 Let $k \in \mathbb{N}$. Then, denoting $p(t) = t(1-t)$, we have

(i) If $k \leq m$, $P_{n,m}((u-t)^k;t) = 0;\quad t \in I$

(ii) $P_{n,m}((u-t)^{m+1};t) = (-1)^m P_n(\prod_{i=0}^{m}(u-t+\frac{i}{n^{1/2}});t)$

$$= (-1)^m \frac{n+1}{p(t)} \{\sum_{r=0}^{m} a_r B_{n+1}((u-t)^{m-r+3};t)\}$$

where a_r's are certain positive constants.

(iii) If $k > m+1$

$$P_{n,m}((u-t)^k;t) = \frac{n+1}{p(t)} \{\sum_{r=0}^{k-1} \frac{b_r}{n^{r/2}} B_{n+1}((u-t)^{k-r+2};t)\}$$

where b_r's are certain constants.

The following theorem of Sinha [61] describes the convergence of the interpolatory modifications:

Theorem 2.2.11 Let $f \in L_p(I)$ $(1 \leq p < \infty)$. Then

$$\lim_{n\to\infty} P_{n,m} f = f,$$

the convergence being in the space $L_p(I)$.

The following two theorems of Sinha [61] give an estimate for the error in the approximation of functions in $C^{2k+2}(I)$ and $C^{m+1}(I)$, respectively, for the operators $P_n(\cdot,k,t)$ and $P_{n,m}(\cdot,t)$.

Theorem 2.2.12 Let $f \in C^{2k+2}(I)$. Then there holds

$$P_n(f,k,t) - f(t) = n^{-(k+1)} \sum_{i=1}^{2k+2} Q(i,k,t)\, f^{(i)}(t) + o(n^{-(k+1)}) \quad (n\to\infty)$$

uniformly in $t \in I$, where $Q(i,k,t)$ are certain polynomials in t.

Theorem 2.2.13 Let $f \in C^{m+1}(I)$. Then

$$P_{n,m}(f;t)-f(t) = \frac{(-1)^m}{(m+1)!}\,\frac{p_{m+1}(t)}{n^{(m+1)/2}}\, f^{(m+1)}(t) + o\left(\frac{1}{n^{(m+1)/2}}\right) (n\to\infty)$$

and

$$P_{n,m+1}(f,t) - f(t) = o\left(\frac{1}{n^{(m+1)/2}}\right) \qquad (n \to \infty)$$

uniformly in $t \in I$, where $p_{m+1}(t)$ is a polynomial in t of degree $\leq m+1$ and $p_{m+1}(t) > 0$ for t in the interior of I.

The following lemma of Sinha [61] can be characterised as Bernstein type inequalities for the operators $P_n(\cdot,t)$ and $P_{n,m}(\cdot,t)$.

Lemma 2.2.14 Let $h \in L_p(I)$ $(1 \leq p < \infty)$ have supp $h \subset I_2$. Then

(1) $$||P_n^{(m+1)}(h;t)||_{L_p(I_2)} \leq M_1 n^{\left(\frac{m+1}{2}\right)} ||h||_{L_p(I_2)}$$

and

(2) $$||P_{n,m}^{(m+1)}(h;t)||_{L_p(I_2)} \leq M_3\, n^{\left(\frac{m+1}{2}\right)} ||h||_{L_p(I_2)}$$

If, in addition, h has $m+1$ derivatives with $h^{(m)} \in A.C\,(I_2)$ and $h^{(m+1)} \in L_p(I_2)$ then

(3) $$||P_n^{(m+1)}(h;t)||_{L_p(I_2)} \leq M_2 ||h^{(m+1)}||_{L_p(I_2)} .$$

and

(4) $$||P_{r,m}^{(m+1)}||_{L_p(I_2)} \leq M_4 ||h^{(m+1)}||_{L_p(I_2)} .$$

Here, M_i, $i=1,2,3,4$ are constants independent of n and h.

Next, we give some technical lemmas of Sinha [61] which are of use in the proofs of our inverse theorems.

Lemma 2.2.15 Let $h \in L_p(I)$ $(1 \leq p < \infty)$ and $i,j \in \mathbb{N}^0$. Then for any fixed $\ell > 0$, there holds

$$||(n+1)\sum_{\nu=0}^{n}\{P_{n\nu}(t)|\frac{\nu}{n}-t|^i \int_{\frac{\nu}{(n+1)}}^{\frac{(\nu+1)}{n+1}} |u-t|^j|\int_t^u |h(w)|dw|du\}||_{L_p(I_2)}$$

$$\leq M\{n^{-(i+j+1)/2}||h||_{L_p(I_1)} + n^{-\ell}||h||_{L_p(I)}\},$$

where M is a constant independent of n and h .

Lemma 2.2.16 Let $i \in \mathbb{N}$ and $h \in L_p(I)$ $(1 \leq p < \infty)$. Then, for any fixed $\ell > 0$, there holds

$$||P_n(|u-t|^i|h(u)|;t)||_{L_p(I_2)}$$

$$\leq M\{\frac{1}{n^{i/2}}||h||_{L_p(I_1)} + \frac{1}{n^{\ell}}||h||_{L_p(I)}\},$$

M being independent of n and h.

Lemma 2.2.17 Let $j,k,s \in \mathbb{N}^0$, $1 \leq p < \infty$ and $h \in L_p(I)$.

Then for any fixed $\ell > 0$ and for all sufficiently large values of n

$$\left\| (n+1)\left\{ \sum_{\nu=0}^{n} p_{n\nu}(t) \left|\frac{\nu}{n} - t\right|^j \left\{ \int_{\frac{\nu}{(n+1)}}^{\frac{(\nu+1)}{(n+1)}} |u-t|^k \times \right.\right.\right.$$
$$\left.\left.\left. \times \left| \int_t^{u+\frac{s}{n^{1/2}}} |h(w)|\,dw \right| du \right\}\right\} \right\|_{L_p(I_2)}$$
$$\leq M\{n^{-(k+j+1)/2} \|h\|_{L_p(I_1)} + n^{-\ell} \|h\|_{L_p(I)}\}$$

where M does not depend on n and h.

We combine the local direct theorems of Sinha [61] for the operators $P_n(\cdot,k,t)$ and $P_{n,m}(\cdot,t)$ in

Theorem 2.2.18 Let $f \in L_p(I)$ $(1 \leq p < \infty)$. Then, for all sufficiently large values of n

(1) $$\|P_n(f,k,t) - f(t)\|_{L_p(I_2)} \leq M_1\{\omega_{2k+2}(f,n^{-1/2},p,I_1) + n^{-(k+1)} \|f\|_{L_p(I)}\}$$

and

(2) $$\|P_{n,m}(f,t) - f(t)\|_{L_p(I_2)} \leq M_2\{\omega_{m+1}(f,n^{-1/2},p,I_1) + n^{-(m+1)} \|f\|_{L_p(I)}\},$$

M_1, M_2 being constants.

We conclude this section by stating the local $O(n^{-\alpha/2})$-inverse theorems of Sinha [61] for the operators $P_n(\cdot,k,t)$ and $P_{n,m}(\cdot,t)$.

Theorem 2.2.19 Let $0 < \alpha < 2k+2$ and $f \in L_p(I)$ $(1 \leq p < \infty)$. Then

$$||P_n(f,k,t) - f(t)||_{L_p(I_1)} = O(n^{-\alpha/2}) \qquad (n \to \infty)$$

implies that

$$\omega_{2k+2}(f,t,p,I_2) = O(t^{\alpha}) \qquad (t \to 0).$$

Theorem 2.2.20 Let $0 < \alpha < m+1$ and $f \in L_p(I)$ $(1 \leq p < \infty)$. Then

$$||P_{n,m}(f,t) - f(t)||_{L_p(I_1)} = O(n^{-\alpha/2}) \qquad (n \to \infty)$$

implies that

$$\omega_{m+1}(f,t,p,I_2) = O(t^{\alpha}) \qquad (t \to 0).$$

Theorems 2.2.19-20 will follow as very special cases of our $O(\varphi)$-inverse theorems 2.3.1 and 2.5.1.

2.3 $O(\varphi)$-inverse theorem for $P_n(\cdot,k,t)$

By theorem 2.2.18, if $\omega_{2k+2}(f,t,p,I_1) = O(\varphi(t))$ $(t \to 0)$, then $||P_n(f,k,t) - f(t)||_{L_p(I_2)} = O(\varphi(n^{-1/2}))$, $(n \to \infty)$. The corresponding inverse theorem is as follows:

Theorem 2.3.1 Let $\varphi \in \Phi_{2k+2}$ and $f \in L_p[0,1]$ $(1 \leq p < \infty)$. Then

$$(2.3.1) \qquad ||P_n(f,k,t) - f(t)||_{L_p(I_1)} = O(\varphi(n^{-1/2})) \qquad (n \to \infty)$$

implies that

$$(2.3.2) \qquad \omega_{2k+2}(f,t,p,I_2) = O(\varphi(t)) \qquad (t \to 0).$$

Proof: Let x_i, y_i's, g and $\bar{f}$ be defined as in the proof of Theorem 1.5.1. Then

$$\|\Delta_\gamma^{2k+2}\bar{f}(t)\|_{L_p[x_3,y_3]} \leq \|\Delta_\gamma^{2k+2}\{\bar{f}(t)-P_n(\bar{f},k,t)\}\|_{L_p[x_3,y_3]}$$

$$+\|\Delta_\gamma^{2k+2}P_n(\bar{f},k,t)\|_{L_p[x_3,y_3]}.$$

By $2k+2$ applications of Jensen's inequality

$$\left|\int_0^\gamma\cdots\int_0^\gamma P_n^{(2k+2)}(\bar{f},k,t+\sum_{i=1}^{2k+2}z_i)dz_1\cdots dz_{2k+2}\right|^p$$

$$\leq \gamma^{(2k+2)(p-1)}\{\int_0^\gamma\cdots\int_0^\gamma |P_n^{(2k+2)}(\bar{f},k,t+\sum_{i=1}^{2k+2}z_i)|^p$$

$$dz_1\cdots dz_{2k+2}\}.$$

Now, using Fubini's theorem, for all γ sufficiently small,

$$\int_{x_3}^{y_3}\left|\int_0^\gamma\cdots\int_0^\gamma P_n^{(2k+2)}(\bar{f},k,t+\sum_{i=1}^{2k+2}z_i)dz_1\cdots dz_{2k+2}\right|^p dt$$

$$\leq \gamma^{(2k+2)p}\|P_n^{(2k+2)}(\bar{f},k,t)\|^p_{L_p[x_3',y_3']},$$

where $x_3' = x_3$ and $y_3' = y_3 + (2k+2)\nu$.

Thus, by lemma 0.6.1 , for all η sufficiently small (as in the proof of the theorem 1.5.1.),

$$\|\Delta_\gamma^{2k+2}P_n(\bar{f},k,t)\|_{L_p[x_3,y_3]} \leq \gamma^{2k+2}\|P_n^{(2k+2)}(\bar{f},k,t)\|_{L_p[x_3',y_3']}$$

$$\leq \gamma^{2k+2}\{\|P_n^{(2k+2)}(\bar{f}-\bar{f}_{\eta,2k+2},k,t)\|_{L_p[x_3',y_3']} +$$

$$+ \| P_n^{(2k+2)}(\bar{f}_{\eta,2k+2},k,t)\|_{L_p[x_3',y_3']} \}$$

$$\leq M_1 \gamma^{2k+2}\{ n^{k+1} \|\bar{f}-\bar{f}_{\eta,2k+2}\|_{L_p[x_3',y_3']}$$

$$+ \|\bar{f}_{\eta,2k+2}\|_{L_p[x_3',y_3']}\} ,$$

by parts 1 and 3 of lemma 2.2.14. Now, applying lemma 0.6.5

$$(2.3.3) \quad \| \Delta_\gamma^{2k+2} P_n(\bar{f},k,t)\|_{L_p[x_3,y_3]}$$

$$\leq M_2 \gamma^{2k+2}(n^{k+1} + \frac{1}{\eta^{2k+2}}) \, \omega_{2k+2}(\bar{f},\eta,p,[x_3,y_3]).$$

The next major step is to show that

$$(2.3.4) \quad \| \Delta_\gamma^{2k+2}\{\bar{f}(t)-P_n(\bar{f},k,t)\}\|_{L_p[x_3,y_3]} = O(\varphi(n^{-1/2})) \quad (n\to\infty).$$

For, after having proved (2.3.4), from (2.3.3), we get

$$\| \Delta_\gamma^{2k+2} \bar{f}(t)\|_{L_p[x_3,y_3]} \leq M_3\{\varphi(n^{-1/2})$$

$$+ \gamma^{2k+2}(n^{k+1} + \frac{1}{\eta^{2k+2}}) \, \omega_{2k+2}(\bar{f},\eta,p,[x_3,y_3]\} .$$

Choosing n such that $n \leq \eta^{-2} < 2n$, and taking supremum over $0 < \gamma \leq t$, for all sufficiently small t, we have

$$\omega_{2k+2}(\bar{f},t,p,[x_3,y_3]) \leq M_4\{\varphi(\eta)+(t/\eta)^{2k+2}\omega_{2k+2}(\bar{f},\eta,p,[x_3,y_3])\}$$

which by lemma 1.2.2, gives

$$\omega_{2k+2}(\bar{f},t,p,[x_3,y_3]) = O(\varphi(t)) \qquad (t\to 0) .$$

and the conclusion follows since $\bar{f}(t) = f(t)$ on I_2.

To prove (2.3.4), it suffices to prove that

$$(2.3.5) \qquad ||P_n(\bar{f},k,t) - \bar{f}(t)||_{L_p[x_3,y_3]} = O(\varphi(n^{-1/2})) \quad (n \to \infty).$$

We prove this by an inductive argument as in the case of Theorem 1.5.1.

First, we prove the theorem for all $\varphi \in \Phi_1$. Next, assuming that for some $r \in \mathbb{N}$, such that $1 \le r \le 2k+1$, the theorem holds, for all $\varphi \in \Phi_r$, we prove the theorem for all $\varphi \in \Phi_{r+1}$.

Hence let $\varphi \in \Phi_1$. Then

$$\begin{aligned} ||P_n(fg,k,t) - (fg)(t)||_{L_p[x_3,y_3]} &\le ||P_n((f(u)-f(t))g(t),k,t)||_{L_p[x_3,y_3]} \\ &\quad + ||P_n(f(u)(g(u)-g(t)),k,t)||_{L_p[x_3,y_3]} \\ &= ||g(t)\{P_n(f,k,t)-f(t)\}||_{L_p[x_3,y_3]} \\ &\quad + ||P_n(f(u)(u-t)g'(\xi),k,t)||_{L_p[x_3,y_3]} \end{aligned}$$

for some ξ lying between u and t.

Now, the hypothesis (2.3.1) alongwith lemma 2.2.16, implies that,

$$(2.3.6) \qquad ||P_n(fg,k,t)-(fg)(t)||_{L_p[x_3,y_3]} \le M_5\, \varphi(n^{-1/2}) + M_6 n^{-1/2}.$$

Since, $\varphi \in \Phi_1$, $hK_\varphi(h) \to 0$ as $h \to 0$. Hence,

(2.3.7) $\qquad t/\varphi(t) \to 0 \quad \text{as} \quad t \to 0.$

Combining (2.3.6-7), we get (2.3.5), for all $\varphi \in \Phi_1$. Hence, the theorem is proved for all $\varphi \in \Phi_1$. Next assume that for some $r \in \mathbb{N}$ such that $1 \leq r \leq 2k+1$, the theorem holds, for all $\varphi \in \Phi_r$. Let $\varphi \in \Phi_{r+1}$. Then,

$$\begin{aligned}
&||P_n(fg,k,t) - (fg)(t)||_{L_p[x_3,y_3]} \\
&\leq M_6'\varphi(n^{-1/2}) + ||P_n(f(u)(g(u)-g(t)),k,t)||_{L_p[x_3,y_3]} \\
&\leq M_6'\varphi(n^{-1/2}) + ||P_n((f(u)-f_{\eta,2k+2}(u)) \times \\
&\qquad\qquad \times (g(u) - g(t)), k,t)||_{L_p[x_3,y_3]} \\
&+ ||P_n((f_{\eta,2k+2}(u)-f_{\eta,2k+2}(t))(g(u)-g(t)),k,t)||_{L_p[x_3,y_3]} \\
&+ ||P_n(f_{\eta,2k+2}(t)(g(u)-g(t)),k,t))||_{L_p[x_3,y_3]} \\
&= M_6'\varphi(n^{-1/2}) + z_1+z_2+z_3, \text{ say.}
\end{aligned}$$

We have, for some ξ lying between u and t,

$$\begin{aligned}
z_1 &\leq ||P_n((f(u)-f_{\eta,2k+2}(u))(u-t)g'(\xi),k,t)||_{L_p[x_3,y_3]} \\
&\leq ||g'||_{C(I)} \{\sum_{j=0}^{k} |C(j,k)| \times \\
&\qquad \{||P_{d_jn}(|f(u)-f_{\eta,2k+2}(u)|\ |u-t|;t)||_{L_p[x_3,y_3]}
\end{aligned}$$

Hence, using lemma 2.2.16,

(2.3.8) $\qquad z_1 \leq M_7\{n^{-1/2}||f-f_{\eta,2k+2}||_{L_p[x_2,y_2]}+n^{-(k+1)}||f||_{L_p(I)}\}$

Again, for some ξ lying between u and t

$$(f_{\eta,2k+2}(u)-f_{\eta,2k+2}(t))(g(u)-g(t))$$

$$= \{\sum_{i=1}^{2k+1} \frac{(u-t)^i}{i!} f^{(i)}_{\eta,2k+2}(t) + \frac{1}{(2k+1)!}\int_t^u (u-w)^{2k+1} f^{(2k+2)}_{\eta,2k+2}(w)\,dw\} \times$$

$$\times \{\sum_{i=1}^{2k} \frac{(u-t)^i}{i!} g^{(i)}(t) + \frac{(u-t)^{2k+1}}{(2k+1)!} g^{(2k+1)}(\xi)\}$$

$$= \sum_{i=1}^{2k+1}\sum_{j=1}^{2k} \frac{f^{(i)}_{\eta,2k+2}(t)}{i!j!} g^{(j)}(t)(u-t)^{i+j}$$

$$+ \frac{g^{(2k+1)}(\xi)}{(2k+1)!}\{\sum_{i=1}^{2k+1}(u-t)^{2k+1+i} f^{(i)}_{\eta,2k+2}(t)\}$$

$$+ \frac{1}{(2k+1)!}\{\sum_{i=1}^{2k}\frac{g^{(i)}(t)}{i!}(u-t)^i \int_t^u (u-w)^{2k+1} f^{(2k+2)}_{\eta,2k+2}(w)\,dw\}$$

$$+ \frac{1}{((2k+1)!)^2}\{g^{(2k+1)}(\xi)(u-t)^{2k+1}\int_t^u (u-w)^{2k+1} f^{(2k+2)}_{\eta,2k+2}(w)\,dw\}$$

Hence

(2.3.9) $$Z_2 \leq R_1 + R_2 + R_3 + R_4, \quad \text{where}$$

$$R_1 = \sum_{i=1}^{2k+1}\sum_{j=1}^{2k} \frac{1}{i!j!}\| f^{(i)}_{\eta,2k+2}(t) g^{(j)}(t)$$

$$P_n((u-t)^{i+j},k,t)\|_{L_p[x_3,y_3]},$$

$$R_2 = \frac{1}{(2k+1)!}\{\sum_{i=1}^{2k+1}\frac{1}{i!}\| f^{(i)}_{\eta,2k+2}(t) \times$$

$$\cdot P_n((u-t)^{2k+1+i}\, g^{(2k+1)}(\xi),k,t)\|_{L_p[x_3,y_3]}\},$$

$$R_3 = \frac{1}{(2k+1)!} \| P_n\Big(\sum_{i=1}^{2k} g^{(i)}(t)(u-t)^i \Big(\int_t^u (u-w)^{2k+1} \times$$

$$\times \Big(\int_t^u (u-w)^{2k+1} f_{\eta,2k+2}^{(2k+2)}(w)\,dw\Big), k, t\Big) \|_{L_p[x_3,y_3]}$$

and

$$R_4 = \frac{1}{((2k+1)!)^2} \| P_n\Big(g^{(2k+1)}(\xi)(u-t)^{2k+1} \times$$

$$\times \Big(\int_t^u (u-w)^{2k+1} f_{\eta,2k+2}^{(2k+2)}(w)\,dw\Big), k, t\Big) \|_{L_p[x_3,y_3]} .$$

Since $\sum_{j=0}^{k} C(j,k) d_j^{-m} = 0$, $m=1,2,\ldots,k$, it follows from corollary 2.2.4 and lemma 0.6.5 , that

$$(2.3.10) \quad R_1 \le \frac{M_8}{n^{k+1}} \{ \| f_{\eta,2k+2}^{(2k+1)} \|_{L_p[x_3,y_3]} + \| f_{\eta,2k+2} \|_{L_p[x_3,y_3]} \} .$$

Similarly,

$$(2.3.11) \quad R_2 \le \frac{M_9}{n^{k+1}} \{ \| f_{\eta,2k+2}^{(2k+1)} \|_{L_p[x_3,y_3]} + \| f_{\eta,2k+2} \|_{L_p[x_3,y_3]} \} .$$

By lemma 2.2.15 upon taking $\ell = 2k+2$,

$$(2.3.13) \quad R_3 \le M_{10} \{ n^{-(k+3/2)} \| f_{\eta,2k+2}^{(2k+2)} \|_{L_p[x_2,y_2]} + n^{-(2k+2)} \| f_{\eta,2k+2}^{(2k+2)} \|_{L_p(I)} \}$$

and

$$(2.3.14) \quad R_4 \le M_{11} \{ n^{-(2k+2)} \| f_{\eta,2k+2}^{(2k+2)} \|_{L_p[x_2,y_2]} + n^{-(2k+2)} \| f_{\eta,2k+2}^{(2k+2)} \|_{L_p(I)} \} .$$

Choosing n such that $n < \eta^{-2} \leq 2n$, it follows from (2.3.10–11) and lemma 0.6.5 that

$$R_1, R_2 \leq M_{12}\{n^{-1/2}\omega_{2k+1}(f,n^{-1/2},p,[x_1,y_1]) + n^{-(k+1)}||f||_{L_p(I)}\},$$

$$R_3, R_4 \leq M_{13}\{n^{-1/2}\omega_{2k+2}(f,n^{-1/2},p,[x_1,y_1])+n^{-(k+1)}||f||_{L_p(I)}\}.$$

and hence from 2.3.9,

$$(2.3.15) \quad Z_2 \leq M_{14}\{n^{-1/2}\omega_{2k+2}(f,n^{-1/2},p,[x_1,y_1]) + n^{-1/2}\omega_{2k+1}(f,n^{-1/2},p,[x_1,y_1]) + n^{-(k+1)}||f||_{L_p(I)}\}.$$

Also, from theorem 2.2.12 and lemma 0.6.5

$$(2.3.16) \quad Z_3 \leq \frac{M_{15}}{n^{k+1}}||f||_{L_p(I)}.$$

Hence, from (2.3.8), (2.3.14–15), and using again lemma 0.6.5 we get

$$(2.3.17) \quad Z_1+Z_2+Z_3 \leq M_{16}\{n^{-1/2}\omega_{2k+2}(f,n^{-1/2},p,[x_1,y_1]) + n^{-1/2}\omega_{2k+1}(f,n^{-1/2},p,[x_1,y_1]) + n^{-(k+1)}||f||_{L_p(I)}\}.$$

Now, defining $\varphi^*(t) = \frac{\varphi(t)}{t}$, we observe that $\varphi^* \in \Phi_r \subset \Phi_{2k+1}$.

Also, we observe that

$$||P_n(f,k,t)-f(t)||_{L_p(I_1)} = O(\varphi^*(n^{-1/2})) \quad (n \to \infty).$$

Hence, by induction hypothesis,

(2.3.18) $\quad \omega_{2k+2}(f,t,p,[x_1,y_1]) = O(\varphi^*(t)) \qquad (t \to 0)$

which, from the fact that $\varphi^* \in \Phi_{2k+1}$ and lemma 1.2.4, implies that

(2.3.19) $\quad \omega_{2k+1}(f,t,p,[x_1,y_1]) = O(\varphi^*(t)) \qquad (t \to 0).$

Hence, from (2.3.17-19) and the definition of φ^*, we get

$$Z_1+Z_2+Z_3 \leq M_{17}\{\varphi(n^{-1/2}) + n^{-(k+1)}\|f\|_{L_p(I)}\}$$

which proves the theorem, since the fact $\frac{t^{2k+2}}{\varphi(t)} \to 0$ follows from the fact that $\varphi \in \Phi_{r+1} \quad \Phi_{2k+2}$.

<u>Corollary</u> 2.3.2 Let $1 \leq m \leq 2k+2$, $\varphi \in \Phi_m$ and $f \in L_p[0,1]$ $(1 \leq p < \infty)$. Then

$$\|P_n(f,k,t) - f(t)\|_{L_p(I_1)} = O(\varphi(n^{-1/2})) \qquad (n \to \infty)$$

implies that

$$\omega_m(f,t,p,I_2) = O(\varphi(t)) \qquad (t \to 0).$$

<u>Proof</u> Follows as in the case of corollary 1.5.4.

<u>Corollary</u> 2.3.3 Theorem 2.2.19

<u>Proof</u> $\quad \varphi(t) = t^{\alpha} \in \Phi_{2k+2}$.

2.4 $o(\varphi)$ inverse theorems for $P_n(\cdot,k,t)$

Theorem 2.2.18 implies that, if

$$\omega_{2k+2}(f,t,p,I_1) = o(\varphi(t)) \quad (t \to 0)$$

Then

$$||P_n(f,k,t) - f(t)||_{L_p(I_2)} = o(\varphi(n^{-1/2})) \quad (n \to \infty).$$

Correspondingly, we prove an inverse theorem for $o(\varphi)$.

<u>Theorem</u> 2.4.1 Let $\varphi \in \Phi_{2k+2}$ and $f \in L_p[0,1] (1 \leq p < \infty)$. Then

$$||P_n(f,k,t) - f(t)||_{L_p(I_1)} = o(\varphi(n^{-1/2})) \quad (n \to \infty)$$

implies that

$$(2.4.1) \qquad \omega_{2k+2}(f,t,p,I_2) = o(\varphi(t)) \quad (t \to 0).$$

<u>Proof</u> Choosing (x_i,y_i), $i=1,2,3,4,5$, $g \in C_0^{2k+2}$ and $fg = \bar{f}$, as in the theorem 1.5.1 and proceeding as in the theorem 2.3.1, for sufficiently small η, $\gamma > 0$, we get

$$(2.4.2) \qquad ||\Delta_\gamma^{2k+2} P_n(\bar{f},k,t)||_{L_p[x_3,y_3]}$$

$$\leq M_1 \gamma^{2k+2}(n^{k+1} + \frac{1}{\eta^{2k+2}}) \omega_{2k+2}(\bar{f},\eta,p,[x_3,y_3]).$$

Next, we shall show that

$$(2.4.3) \qquad ||\Delta_\gamma^{2k+2}\{\bar{f}(t)-P_n(\bar{f},k,t)\}||_{L_p[x_3,y_3]} = o(\varphi(n^{-1/2})) \quad (n\to\infty).$$

After having proved (2.4.3), we define $\psi(t)$ as follows:

$$(2.4.4) \qquad \psi(x) = \begin{cases} ||\Delta_\gamma^{2k+2}\{\bar{f}(t)-P_n(f,k,t)\}||_{L_p[x_3,y_3]} & \text{if } x = n^{-1/2} \text{ for any } n \in \mathbb{N} \\ \psi(n^{-1/2}) & \text{if } x \in ((n+1)^{-1/2}, n^{-1/2}) \text{ for any } n \in \mathbb{N} \end{cases}$$

Clearly $\psi(t) = o(\varphi(t))$ and hence combining (2.4.2-4), we get

$$||\Delta_{\gamma}^{2k+2} f(t)||_{L_p[x_3,y_3]} \leq M_2 \{\psi(n^{-1/2}) + \gamma^{2k+2}(n^{k+1} + \frac{1}{\eta^{2k+2}}) \times$$
$$\times\, \omega_{2k+2}(\bar{f},\eta,p,[x_3,y_3])\},$$

Since this holds for all t sufficiently small and $\gamma \leq t$, choosing n such that $n \leq \eta^{-2} < n+1$, we get

$$\omega_{2k+2}(\bar{f},t,p,[x_3,y_3])$$
$$\leq M_3\{\psi(\eta) + (\frac{t}{\eta})^{2k+2} \omega_{2k+2}(\bar{f},\eta,p,[x_3,y_3])\},$$

which, by lemma 1.2.3, implies

$$\omega_{2k+2}(\bar{f},t,p,[x_3,y_3]) = o(\varphi(t)) \qquad (t \to 0)$$

and since $\bar{f} = f$ on I_2, we get

$$\omega_{2k+2}(f,t,p,I_2) = o(\varphi(t)) \qquad (t \to 0).$$

Hence, to complete the proof of the theorem, we are left to show that

$$(2.4.5) \quad ||P_n(\bar{f},k,t) - \bar{f}(t)||_{L_p[x_3,y_3]} = o(\varphi(n^{-1/2})) \quad (n \to \infty),$$

We prove this by induction as follows :

First, we prove the theorem for all $\varphi \in \Phi_1$. Later, assuming the theorem for all $\varphi \in \Phi_r$ (for some r such that $1 \leq r \leq 2k+1$), we prove the theorem for $\varphi \in \Phi_{r+1}$.

Let $\varphi \varepsilon \Phi_1$. Then, proceeding as in the proof of theorem 2.3.1 and using the fact that $t/\varphi(t) \to 0$ as $t \to 0$,

$$||P_n(fg,k,t) - (fg)(t)||_{L_p[x_3,y_3]} = o(\varphi(n^{-1/2})) \quad (n \to \infty)$$

and hence the theorem is proved, for all $\varphi \varepsilon \Phi_1$.

Now, assume that for some $r \varepsilon \mathbb{N}$ such that $1 \leq r \leq 2k+1$, the theorem holds for all $\varphi \varepsilon \Phi_r$. Let $\varphi \varepsilon \Phi_{r+1}$. Again, proceeding as in the proof of theorem 2.3.1 and choosing n such that $n \leq \eta^{-2} < n+1$, we get

(2.4.6) $$||P_n(fg,k,t) - (fg)(t)||_{L_p[x_3,y_3]} = Z_o+Z_1+Z_2+Z_3$$

where

(2.4.7) $$Z_o = ||g(t)\ \{P_n(f,k,t) - f(t)\}||_{L_p[x_3,y_3]}$$

$$= o(\varphi(n^{-1/2})) \quad (n \to \infty)$$

and Z_1, Z_2, Z_3 are as before and get estimated as follows :

(2.4.8) $$Z_1+Z_2+Z_3 \leq M_2\{n^{-1/2}\ \omega_{2k+2}(f,n^{-1/2},p,[x_1,y_1])$$

$$+ n^{-1/2}\ \omega_{2k+1}(f,n^{-1/2},p,[x_1,y_1]) + n^{-(k+1)}||f||_{L_p(I)}\}.$$

Now $\varphi^* = \dfrac{\varphi(t)}{t} \varepsilon \Phi_r \subset \Phi_{2k+1}$ and we can write

$$||P_n(f,k,t)-f(t)||_{L_p(I_1)} = o(\varphi^*(n^{-1/2})) \quad (n \to \infty).$$

Hence, by induction hypothesis

$$(2.4.9) \qquad \omega_{2k+2}(f,t,p,[x_1,y_1]) = o(\varphi^*(t)) \qquad (t \to 0)$$

and by lemma 1.2.5, we get

$$(2.4.10) \qquad \omega_{2k+1}(f,t,p,[x_1,y_1]) = o(\varphi^*(t)) \qquad (t \to 0).$$

Again, $\varphi \in \Phi_{r+1} \subset \Phi_{2k+2}$ implies that

$$h^{2k+2} K_\varphi(h) \to 0 \quad \text{as } h \to 0$$

which implies that

$$(2.4.11) \qquad t^{2k+2}/\varphi(t) \to 0 \quad \text{as } t \to 0.$$

Hence, combining (2.4.6-11), we get

$$Z_0+Z_1+Z_2+Z_3 = o(\varphi(n^{-1/2})) \qquad (n \to \infty)$$

and the theorem follows.

By theorem 2.4.1 and a recursive use of lemma 1.2.5, we get the following

Corollary 2.4.2. Let $f \in L_p[0,1]$ $(1 \leq p < \infty)$ and $m \in \mathbb{N}$ be such that $1 \leq m \leq 2k+2$. Then, if $\varphi \in \Phi_m$,

$$\|P_n(f,k,t) - f(t)\|_{L_p(I_1)} = o(\varphi(n^{-1/2})) \qquad (n \to \infty)$$

implies that

$$\omega_m(f,t,p,I_2) = o(\varphi(t)) \qquad (t \to 0).$$

2.5 O (φ)-inverse theorem for $P_{n,m}(\cdot,t)$.

Theorem 2.2.18 implies that, if

$$\omega_{m+1}(f,t,p,I_1) = O(\varphi(t)) \qquad (t \to 0),$$

then

$$||P_{n,m}(f,t)-f(t)||_{L_p(I_2)} = O(\varphi(n^{-1/2})) \qquad (n \to \infty).$$

Now, as in the case of $P_n(f,k,t)$, we prove the corresponding local $O(\varphi)$-inverse theorem for $P_{n,m}(\cdot,t)$.

<u>Theorem</u> 2.5.1. Let $\varphi \in \Phi_{m+1}$ and $f \in L_p(I)$ $(1 \le p < \infty)$. Then,

$$||P_{n,m}(f;t) - f(t)||_{L_p(I_1)} = O(\varphi(n^{-1/2})) \qquad (n \to \infty)$$

implies that

$$\omega_{m+1}(f,t,p,I_2) = O(\varphi(t)) \qquad (t \to 0).$$

<u>Proof</u>. Choosing $g \in C_0^{m+1}$, x_i, y_i, $i = 1,2,3,4,5$ and $\bar{f}$ as in the proof of theorem 1.5.1 and proceeding as in proof of theorem 1.5.1, we get

$$||\Delta_\gamma^{m+1} P_{n,m}(\bar{f};t)||_{L_p[x_3,y_3]}$$

$$\le \gamma^{m+1} \{||P_{n,m}^{(m+1)}(\bar{f}-\bar{f}_{\eta,m+1};t)||_{L_p[x_3,y_3]}$$

$$+ ||P_n^{(m+1)}(\bar{f}_{\eta,m+1};t)||_{L_p[x_3,y_3]}\}.$$

Now, for sufficiently small γ, $\eta>0$, by lemmas 2.2.14 and 0.6.5

$$||\Delta_\gamma^{m+1} P_{n,m}(\bar{f};t)||_{L_p[x_3,y_3]}$$

$$\le M_1 \gamma^{m+1}\{n^{(m+1)/2}||\bar{f}-\bar{f}_{\eta,m+1}||_{L_p[x_3,y_3]} + ||\bar{f}_{\eta,m+1}^{(m+1)}||_{L_p[x_3,y_3]}\}$$

$$\le \gamma^{m+1}(n^{\frac{m+1}{2}} + \frac{1}{\eta^{m+1}})\ \omega_{m+1}(\bar{f},\eta,p,[x_3,y_3]) .$$

Again, proceeding as for theorem 2.3.1, we are left to show that

$$(2.5.1)\qquad ||\bar{f}(t)-P_{n,m}(\bar{f};t)||_{L_p[x_3,y_3]} = O(\varphi(n^{-1/2})), \qquad (n\to\infty),$$

which we first prove for $\varphi \in \Phi_1$ and then $\varphi \in \Phi_{r+1}$ assuming the result of the theorem for $\varphi \in \Phi_r$, $1 \leq r \leq m$.

Let $\varphi \in \Phi_1$. Then, for some ξ lying between u and t,

$$||P_{n,m}(fg;t) - (fg)(t)||_{L_p[x_3,y_3]}$$

$$\leq M_3\ \varphi(n^{-1/2}) + ||P_{n,m}(f(u)(u-t)\,g'(\xi);t)||_{L_p[x_3,y_3]}\ .$$

A typical component of

$$Z = P_{n,m}(f(u)(u-t)\ g'(\xi);t)$$

can be written as

$$T_1(t) = c\ n^{(j-r-i)/2} \int_0^1 K(n,t,u)(t-u)^{j-r-i+1} \times$$

$$\times f(u+\frac{k}{n^{1/2}})\ g'(\xi_k)\ du,$$

where $i,j,k,r \in \mathbb{N}'$, $0 \leq j \leq m$, $0 \leq k \leq j$, $0 \leq r \leq j-1$, $i = 0,1$, ξ_k lies between $u + k/n^{1/2}$ and t, c is a constant and $K(n,t,u)$ is given by (2.2.1).

Now, from lemma 2.2.16 and the fact that $t/\varphi(t) \to 0$ as $t \to 0$, we get that

$$||T_1(t)||_{L_p[x_3,y_3]} \leq M_4\ \varphi(n^{-1/2}) \qquad (n\to\infty),$$

which implies that

$$||Z||_{L_p[x_3,y_3]} \leq M_5\ \varphi(n^{-1/2}) \qquad (n \to \infty).$$

Hence the theorem is proved for all $\varphi \in \Phi_1$.

Now, assume that the theorem is true for all $\varphi \in \Phi_r$ $(1 \leq r \leq m)$. Let $\varphi \in \Phi_{r+1}$. Then

$$||P_{n,m}(fg;t)-(fg)(t)||_{L_p[x_3,y_3]} \leq M_3\ \varphi(n^{-1/2}) + Z_1+Z_2+Z_3$$

where

$$Z_1 = ||P_{n,m}((f(u)-f_{\eta,m+1}(u))(g(u)-g(t));t)||_{L_p[x_3,y_3]},$$

$$Z_2 = ||P_{n,m}((f_{\eta,m+1}(u)-f_{\eta,m+1}(t))(g(u)-g(t));t)||_{L_p[x_3,y_3]}$$

and

$$Z_3 = ||P_{n,m}(f_{\eta,m+1}(t)(g(u)-g(t));t)||_{L_p[x_3,y_3]}.$$

By theorem 2.2.13 and lemma 0.6.5, we get

$$(2.5.2) \qquad Z_3 \leq M_6\ n^{-(\frac{m+1}{2})}\ ||f||_{L_p(I)}.$$

For some ξ lying between u and t,

$$(2.5.3) \quad Z_1 = ||P_{n,m}((f(u)-f_{\eta,m+1}(u))(u-t)\,g'(\xi);t)||_{L_p[x_3,y_3]}.$$

A typical component of

$$P_{n,m}((f(u)-f_{\eta,m+1}(u))(u-t)\ g'(\xi);t)$$

can be written as

$$T_2(t) = c\ n^{(j-r-i)/2} \int_0^1 K(n,t,u)(t-u)^{j-r-i+1} \times$$

$$\times\ (f(u+\frac{k}{n^{1/2}})-f_{\eta,m+1}(u+\frac{k}{n^{1/2}}))\ g'(\xi_k)\ du,$$

where $i,j,k,r \in \mathbb{N}^{0}$, $0 \le j \le m$, $0 \le k \le j$, $0 \le r \le j-1$, $i = 0,1$, ξ_k lies between $u + \frac{k}{n^{1/2}}$ and t, c is a scalar and $K(n,t,u)$ is given by (2.2.1).

Proceeding as for $T_1(t)$, we get

$$||T_2(t)||_{L_p[x_3,y_3]} \le M_1\{n^{-1/2}||f-f_{\eta,m+1}||_{L_p[x_2,y_2]}$$
$$+ n^{-\ell}||f-f_{\eta,m+1}||_{L_p(I)}\}$$

and thus from (2.5.3),

$$Z_1 \le M_8\{n^{-1/2}||f-f_{\eta,m+1}||_{L_p[x_2,y_2]} + n^{-\ell}||f-f_{\eta,m+1}||_{L_p(I)}\}.$$

Applying lemma 0.6.5

$$(2.5.4) \qquad Z_1 \le M_9\{n^{-1/2}\omega_{m+1}(f,\eta,p,[x_1,y_1]) + n^{-\ell}||f||_{L_p(I)}\}.$$

For some ξ lying between u and t,

$$(2.5.5) \qquad Z_2 \le R_1+R_2+R_3+R_4$$

where

$$R_1 = \sum_{i=1}^{m}\sum_{j=1}^{m-1}\frac{1}{i!\,j!}||f^{(i)}_{\eta,m+1}(t)\, g^{(j)}(t)P_{n,m}((u-t)^{i+j};t)||_{L_p[x_3,y_3]},$$

$$R_2 = \frac{1}{m!}\{\sum_{i=1}^{m}||f^{(i)}_{\eta,m+1}(t)P_{n,m}((u-t)^{m+i}g^{(m)}(\xi);t)||_{L_p[x_3,y_3]}\},$$

$$R_3 = \frac{1}{m!}\{\sum_{i=1}^{m-1}\frac{1}{i!}||g^{(i)}(t)\{P_{n,m}((u-t)^{i}\times$$
$$\times(\int_t^u(u-w)^m f^{(m+1)}_{\eta,m+1}(w)\,dw;t)\}||_{L_p[x_3,y_3]}\}$$

and

$$R_4 = \frac{1}{(m!)^2} || P_{n,m}(g^{(m)}(\xi)(u-t)^m \int_t^u (u-w)^m f^{(m+1)}_{\eta,m+1}(w)\,dw;t) ||_{L_p[x_3,y_3]}.$$

After expanding Δ^j and $\prod_{i=0}^{j-1}$ a typical component of

$$P_{n,m}(g^{(m)}(\xi)(u-t)^k(\int_t^u (u-w) f^{(m+1)}_{\eta,m+1}(w)\,dw);t)$$

can be written as

$$(2.5.6) \quad T_3(t) = c\, n^{(\theta-k)/2} \int_0^1 K(n,t,u)(u-t)^\theta g^{(m)}(\xi_{r_2}) \times$$

$$\times \{\int_t^{u+\frac{r_2}{n^{1/2}}} (u-w+\frac{r_2}{n^{1/2}})^m f^{(m+1)}_{\eta,m+1}(w)\,dw\}\, du,$$

where $T_3(t) = T_3(t,j,r_1,r_2,r_3)$, $\theta = j+r_3-r_1$, $0 \le j \le m$, $0 \le r_1 \le j-1$, $0 \le r_2 \le j$, $0 \le r_3 \le k$, ξ_{r_2} lies between $u+\frac{r_2}{n^{1/2}}$ and t, c is a constant and $K(n,t,u)$ is given by (2.2.1).

Let $X(u)$ be the characteristic function of $[c,d]$ where $x_2 < c < x_3 < y_3 < d < y_2$. Then

$$(2.5.7) \quad T_3(t) = T_{31}(t) + T_{32}(t)$$

where

$$T_{31}(t) = c\, n^{\frac{(\theta-k)}{2}} \{\int_0^1 X(u) K(n,t,u)(u-t)^\theta g^{(m)}(\xi_{r_2}) \times$$

$$\times \{\int_t^{u+\frac{r_2}{n^{1/2}}} (u-w+\frac{r_2}{n^{1/2}})^m f^{(m+1)}_{\eta,m+1}(w)\,dw\}du\},$$

and

$$T_{32}(t) = c\, n^{(\theta-k)/2}\{\int_0^1 (1-X(u))\; K(n,t,u)\; (u-t)^\theta\; g^{(m)}\; (\xi_{r_2}) \times$$

$$\times \{\int_t^{u+\frac{r_2}{n^{1/2}}} (u-w+\frac{r_2}{n^{1/2}})^m\; f^{(m+1)}_{\eta,m+1}(w)\,dw\}\; du\}.$$

It is clear from lemma 2.2.15 that, for sufficiently large values of n

$$||T_{31}(t)||_{L_p[x_3,y_3]} \le \frac{M_{10}}{n^{(k+m+1)/2}}\; ||f^{(m+1)}_{\eta,m+1}||_{L_p[x_2,y_2]}.$$

Proceeding as in the proof of lemma 2.2.19, for any $\ell > 0$ and for all $t \in [x_3,y_3]$,

$$|T_{32}(t)| \le \frac{M_{11}}{n^\ell}\; ||f^{(m+1)}_{\eta,m+1}||_{L_p(I)}$$

and hence

$$||T_{32}(t)||_{L_p[x_3,y_3]} \le \frac{M_{12}}{n^\ell}\; ||f^{(m+1)}_{\eta,m+1}||_{L_p(I)}.$$

Thus, from (2.5.6-7), we get

$$||T_3(t)||_{L_p[x_3,y_3]} \le M\{n^{-(k+m+1)/2}\; ||f^{(m+1)}_{\eta,m+1}||_{L_p[x_2,y_2]}$$

$$+\; n^{-\ell}\; ||f^{(m+1)}_{\eta,m+1}||_{L_p(I)}\},$$

which, by lemma 0.6.5, implies

$$||P_{n,m}(g^{(m)}(\xi)(u-t)^k(\int_t^u (u-w)^m\; f^{(m+1)}_{\eta,m+1}(w)\,dw)\,;t)||_{L_p[x_3,y_3]}$$

$$\le M_{13}\{n^{-(k+m+1)/2}\; \eta^{-(m+1)}\; \omega_{m+1}(f,\eta,p,[x_1,y_1])$$

$$+\; n^{-\ell}\; \eta^{-(m+1)}\; ||f||_{L_p(I)}\}.$$

Thus,

$$(2.5.8)\quad R_3 \le M_{14}\{n^{-(\frac{m+2}{2})}\ \eta^{-(m+1)}\omega_{m+1}(f,\eta,p,[x_1,y_1])$$

$$+ n^{-\ell}\eta^{-(m+1)}\ ||f||_{L_p(I)}\}$$

and

$$(2.5.9)\quad R_4 \le M_{15}\{n^{-(\frac{2m+1}{2})}\ \eta^{-(2m+1)}\ \omega_{m+1}(f,\eta,p,[x_1,y_1])$$

$$+ n^{-\ell}\eta^{-(m+1)}\ ||f||_{L_p(I)}\}.$$

By lemma 2.2.10 and lemma 2.2.2,

$$R_1 \le M_{16}\{\sum_{i=1}^{m}\ \sum_{j=1}^{m-1}\ \frac{1}{n^{(i+1)/2}}\ ||f^{(i)}_{\eta,m+1}||_{L_p[x_3,y_3]}\},$$

where the summation is taken only over those i,j's which satisfy $i+j > m$. Thus, in conjunction with lemmas 0.6.3 and 0.6.5

$$(2.5.10)\qquad R_1 \le \frac{M_{17}}{n^{(\frac{m+1}{2})}}\ \{\frac{1}{\eta^m}\,\omega_m(f,\eta,p,[x_2,y_2]) + ||f||_{L_p(I)}\}\ .$$

A typical term in

$$P_{n,m}(g^{(m)}(\xi)\ \ (u-t)^{m+i},t)$$

can be written as

$$T_4(t) = c\ n^{(j+k-r-m-i)/2}\int_0^1 K(n,t,u)\ \ g^{(m)}(\xi_s)\ \times$$

$$\times\ (u-t)^{j+k-r}\ du\ ,$$

where $0 \le j \le m$, $0 \le r \le j-1$, $0 \le s \le j$, $0 \le k \le m+i$, ξ_s lies between $u + \frac{s}{n^{1/2}}$ and t, c is a constant and $K(n,t,u)$ is given by (2.2.1).

Now, from corollary 2.2.5, for all $t \varepsilon [x_3, y_3]$, we obtain

$$|T_4(t)| \leq \frac{M_{18}}{n^{(m+i)/2}} .$$

Consequently,

$$R_2 \leq M_{19} \{ \sum_{i=1}^{m} n^{-(m+i)/2} \|f_{\eta,m+1}^{(i)}\|_{L_p[x_3,y_3]} \}.$$

Again, applying lemmas 0.6.3 and 0.6.5

$$(2.5.11) \qquad R_2 \leq \frac{M_{20}}{n^{(m+1)/2}} \{\frac{1}{\eta^m} \omega_m(f,\eta,p,[x_2,y_2] + \|f\|_{L_p(I)}\}.$$

Choosing n such that $n < \eta^{-2} \leq 2n$ and taking $\ell = m+1$, from (2.5.2), (2.5.4-5) and (2.5.8-11), we get

$$(2.5.12) \qquad Z_1+Z_2+Z_3 \leq M_{21}\{n^{-1/2} \omega_{m+1}(f,n^{-1/2},p,[x_1,y_1])$$

$$+ n^{-1/2} \omega_m(f,n^{-1/2},p,[x_1,y_1]) + n^{-1/2} \|f\|_{L_p(I)}\}.$$

Let $\varphi^*(t) = \frac{\varphi(t)}{t}$. Then $\varphi^* \varepsilon \Phi_r \subset \Phi_m$ and

$$\|P_{n,m}(f,t)-f(t)\|_{L_p(I_1)} = O(\varphi^*(n^{-1/2})) \qquad (n \to \infty).$$

Hence, by induction hypothesis,

$$(2.5.13) \qquad \omega_{m+1}(f,t,p,[x_1,y_1]) = O(\varphi^*(t)) \qquad (t \to 0),$$

which, by lemma 1.2.4, implies that

$$(2.5.14) \qquad \omega_m(f,t,p,[x_1,y_1]) = O(\varphi^*(t)) \qquad (t \to 0).$$

Hence, from (2.5.12-14), we get that

$$(2.5.15) \qquad Z_1+Z_2+Z_3 \leq M_{22} \{n^{-1/2} \varphi^*(n^{-1/2}) + n^{-1/2}\varphi^*(n^{-1/2}) + n^{-(\frac{m+1}{2})} ||f||_{L_p(I)}\}.$$

Now, since $\varphi \varepsilon \Phi_{r+1} \subset \Phi_{m+1}$, we have

$$h^{m+1} \cdot K_\varphi(h) \to 0 \quad \text{as} \quad h \to 0$$

which implies that

$$(2.5.16) \qquad t^{m+1}/\varphi(t) \to 0 \text{ as } t \to 0.$$

Hence, from (2.5.15-16), we get that

$$Z_1+Z_2+Z_3 \leq M_{23} \varphi(n^{-2/2})$$

and hence the theorem is proved.

<u>Corollary</u> 2.5.2. Let $f \varepsilon L_p[0,1]$ $(1 \leq p < \infty)$, $r \varepsilon \mathbb{N}$ be such that $1 \leq r \leq m+1$ and $\varphi \varepsilon \Phi_r$. Then

$$||P_{n,m}(f;t) - f(t)||_{L_p(I_1)} = O(\varphi(n^{-1/2})) \qquad (n \to \infty)$$

implies that

$$\omega_r(f,t,p,I_2) = O(\varphi(t)) \qquad (t \to 0).$$

<u>Proof</u>. Follows along the proof of corollary 1.5.4.

Since $\varphi(t) = t^\alpha \varepsilon \Phi_{m+1}$, for any $0 < \alpha < m+1$, we get

<u>Corollary</u> 2.5.3. Theorem 2.2.20.

2.6 <u>$o(\varphi)$-inverse theorem for $P_{n,m}(\cdot,t)$</u>

Theorem 2.2.18 implies that, if

$$\omega_{m+1}(f,t,p,I_1) = o(\varphi(t)) \qquad (t \to 0),$$

then $\|P_{n,m}(f;t) - f(t)\|_{L_p(I_2)} = o(\varphi(n^{-1/2}))$ $(n \to \infty)$.

The corresponding $o(\varphi)$-inverse result is

Theorem 2.6.1. Let $f \in L_p[0,1]$ $(1 \le p < \infty)$ and $\varphi \in \Phi_{m+1}$. Then

$$\|P_{n,m}(f;t) - f(t)\|_{L_p(I_1)} = o(\varphi(n^{-1/2})) \qquad (n \to \infty)$$

implies that

$$(2.6.1) \qquad \omega_{m+1}(f,t,p,I_2) = o(\varphi(t)) \qquad (t \to 0).$$

Proof. Choosing (x_i, y_i), $i = 1$ to 5, $g \in C_0^{2k+2}$ and $\tilde{f}$ as in theorem 1.5.1 and proceeding as in the proof of the same theorem, for sufficiently small $\eta, \gamma > 0$, we get

$$(2.6.2) \qquad \|\Delta_\gamma^{m+1} P_{n,m}(\bar{f},t)\|_{L_p[x_3,y_3]}$$

$$\le M_1 \gamma^{m+1}\left(n^{\frac{m+1}{2}} + \frac{1}{\eta^{m+1}}\right) \omega_{m+1}(\bar{f},\eta,p,[x_3,y_3]).$$

Next step is to prove that

$$(2.6.3) \qquad \|\Delta_\gamma^{m+1} \{\bar{f}(t) - P_{n,m}(\bar{f},t)\}\|_{L_p[x_3,y_3]} = o(\varphi(n^{-1/2}))\,(n \to \infty).$$

After having proved (2.6.3), we define $\psi(t)$ by

$$(2.6.4) \qquad \psi(t) = \begin{cases} \|\Delta_\gamma^{m+1}\{\bar{f}(t)-P_{n,m}(\bar{f},t)\}\|_{L_p[x_3,y_3]} & \text{if } x = n^{-1/2} \\ \psi(n^{-1/2}) & \text{if } x \in ((n+1)^{-1/2}, n^{-1/2}) \text{ for } n = 1,2,\ldots \end{cases}$$

Clearly $\psi(t) = o(\varphi(t))$ $(t \to 0)$ and

$$||\Delta_\gamma^{m+1}\ \bar{f}(t)||_{L_p[x_3,y_3]} \leq M_2\ \{\psi(n^{-1/2})$$
$$+\ \gamma^{m+1}(n^{\frac{m+1}{2}} + \frac{1}{\eta^{m+1}})\ \omega_{m+1}(\bar{f},\eta,p,[x_3,y_3])\ .$$

Since this holds for all sufficiently small γ, choosing n such that $n \leq \eta^{-2} < n+1$, we get

$$\omega_{m+1}(\bar{f},t,p,[x_3,y_3]) \leq M_3\{\psi(\eta)+(t/\eta)^{m+1}\ \omega_{m+1}(\bar{f},t,p,[x_3,y_3])\}\ ,$$

which by lemma 1.2.5, implies that

$$\omega_{m+1}(\bar{f},t,p,[x_3,y_3]) = o(\varphi(t)) \quad (r \to 0).$$

The conclusion (2.6.1) follows since $\bar{f}(t) = f(t)$ on I_2.

Now we prove (2.6.3) by an induction, as before.

Let $\varphi \in \Phi_1$. Then, for some ξ lying between u and t,

$$T = ||P_{n,m}(fg;t) - (fg)(t)||_{L_p[x_3,y_3]}$$
$$\leq ||P_{n,m}((f(u)-f(t))\ g(t);t)||_{L_p[x_3,y_3]}$$
$$+\ ||P_{n,m}(f(u)(g(u)-g(t));t)||_{L_p[x_3,y_3]}$$
$$= ||g(t)\ \{P_{n,m}(f;t) - f(t)\}||_{L_p[x_3,y_3]}$$
$$+\ ||P_{n,m}(f(u)\ (u-t)g'(\xi);t)||_{L_p[x_3,y_3]}\ .$$

Thus,

$$T = o(\varphi(n^{-1/2})) + ||P_{n,m}(f(u)(u-t)g'(\xi);t)||_{L_p[x_3,y_3]} \tag{2.6.5}$$

$$\|P_{n,m}(f;t) - f(t)\|_{L_p(I_1)} = o(\varphi(n^{-1/2})) \quad (n \to \infty)$$

implies that

$$\omega_r(f,t,p,I_2) = o(\varphi(t)) \quad (t \to 0).$$

CHAPTER III

φ-INVERSE THEOREMS FOR LINEAR COMBINATIONS OF GENERALISED MÜLLER'S OPERATORS T_λ

3.1 Introduction.

In this chapter we study the linear combinations $T_\lambda(\cdot,k,t)$ of the generalised Müller's operators $T_\lambda(\cdot,t)$. These were first introduced and studied by Kunwar [32] and include several well-known operators as particular cases. The operators T_λ are defined as follows :

A non negative measurable function on $\mathbb{R}^+$ is said to be an admissible kernel function if

(1) $G(u)$ is continuous at $u = 1$,

(2) for any $\delta > 0$, $\sup\limits_{|u-1|>\delta} G(u) < G(1)$

and(3) there exist $\theta_1,\theta_2 > 0$ such that $(u^{-\theta_1} + u^{\theta_2})\, G(u)$ is bounded on $\mathbb{R}^+$.

We denote the class of all admissible kernels by $T(\mathbb{R}^+)$. By $T_\infty(\mathbb{R}^+)$, we denote the class of all $G \in T(\mathbb{R}^+)$ which are infinite times differentiable in a neighbourhood of '1' and for which $G''(1) \neq 0$, Also, for any $\delta > 0$, T_δ denotes the class of all $G \in T_\infty(\mathbb{R}^+)$ such that supp $G \subset (1-\delta, 1+\delta)$ and G is infinite times differentiable on $(1-\delta, 1+\delta)$.

Let $G \in T(\mathbb{R}^+)$ and $\alpha \in \mathbb{R}$. Then, for λ, $t \in \mathbb{R}^+$, we define

$$(3.1.1)\qquad T_\lambda(f;t) = T_\lambda^{G,\alpha}(f;t) = \frac{t^{\alpha-1}}{a(\lambda)} \int_0^\infty u^{-\alpha}\, G^\lambda(t/u)\, f(u)\, du,$$

where

(3.1.2) $$a(\lambda) = \int_0^\infty u^{\alpha-2}\, G^\lambda(u)\, du,$$

whenever the above integrals exist.

Our primary interest in this chapter is to deal with $T_\lambda f$, with the kernel $G \in T_\infty(\mathbb{R}^+)$ and $f \in L_p(\mathbb{R}^+)$ $(1 \leq p < \infty)$. For $f \in L_p(\mathbb{R}^+)$, for all λ sufficiently large, $T_\lambda f \in L_p(\mathbb{R}^+)$.

The following well-known operators are particular cases of the operators T_λ.

1. The Gamma operator G_n of Muller [43] defined by

$$G_n(f;t) = \frac{t^{n+1}}{n!} \int_0^\infty u^n e^{-ut} f(\tfrac{n}{u})\, du,$$

is a particular case of T_λ with kernel ue^{-u}, $\alpha = 2$ and $\lambda = n$.

2. With kernel $u^{-1} e^{-u^{-1}}$, $\lambda = n$ and $\alpha = 1$, the operators T_λ become the modified Post-Widder operators [39] defined by,

$$S_n(f;t) = \frac{1}{(n-1)!} (\tfrac{n}{t})^n \int_0^\infty e^{-\frac{nu}{t}} u^{n-1} f(u)\, du.$$

3. Taking the kernel as in (2) above, $\alpha = 0$ and $\lambda = k$, the operator T_λ (acting on φ) gives rise to the original Post-Widder operator $L_{k,t}$ [66] , defined by

$$L_{k,t}\,[f(x)] = \frac{1}{k!} (\tfrac{k}{t})^{k+1} \int_0^\infty e^{-\frac{nu}{t}} u^k \varphi(u)\, du,$$

where

$$f(x) = \int_0^\infty e^{-xu} \varphi(u)\, du,$$

is the Laplace transform of φ .

4. Indeed, the above three operators can be obtained from L_λ by choosing, respectively, $p = 1$, $\alpha = 2$; $p = -1$, $\alpha = 1$ and $p = -1$, $\alpha = 0$, where L_λ is defined by

$$L_\lambda(f;t) = \frac{p\lambda^{(\lambda + \frac{\alpha-1}{p})} t^{(\lambda p + \alpha - 1)}}{\Gamma(\lambda + \frac{\alpha-1}{p})} \int_0^\infty u^{-\lambda p - \alpha} e^{-\lambda (t/u)^p} f(u)\,du,$$

where $p \in \mathbb{R}-\{0\}$, $\alpha \in \mathbb{R}$, $\lambda > 0$. The kernel, here, is $u^p e^{-u^p}$.

5. With kernel $e^{-\frac{\log u}{\sqrt{2\sigma}}}$, $\alpha = 3/2$ and $\lambda = 1/t$, the operators T_λ reduce to the operators U_t [41] defined by

$$U_t(f;x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{-\infty}^{\infty} f(x\, e^y) \times$$

$$\times \exp\left[-\frac{1}{2\sigma^2 t}\left(y + \frac{t\sigma}{2}\right)^2\right] dy, \quad (t > 0).$$

We can easily see that in all the above, the kernel belongs to $T_\infty(\mathbb{R}^+)$. It may also be remarked here that the operators T_λ share with the operators U_n of Chapter one the structural property of essentially being generated by powers of a kernel.

Kunwar [32,33] obtained an asymptotic formula of Voronovskaya type for T_λ under the assumption of the existence of $G'''(1)$ with $G''(1) \neq 0$. Under the same assumption, he determined the Lipschitz-Nikolskii constants of the operators T_λ. Later he studied the simultaneous approximation of derivatives of f by the derivative of T_λ f. Also, he obtained

local $O(\lambda^{-\alpha/2})$-inverse theorem $(0 < \alpha < 2k+2)$ and also the saturation theorem for the linear combinations $T_\lambda(\cdot,k,t)$ (defined below) in sup-norm case.

Recently, Winslin [67] has obtained global direct, inverse and saturation theorems for the linear combinations $T_\lambda(\cdot,k,t)$ of the operators $T_\lambda(\cdot,t)$ in L_p-norm $(1 \leq p < \infty)$. His inverse theorems also correspond to the order $O(\lambda^{-\alpha/2})$ $(0 < \alpha < 2k+2)$.

The linear combinations $T_\lambda(\cdot,k,t)$ are defined as follows :

Let $d_0,\dots,d_k$ be k+1 positive numbers. Then the linear combinations $T_\lambda(\cdot,k,t)$ of the operators $T_\lambda(\cdot,t)$ are defined by

$$T_\lambda(\cdot,k,t) = \sum_{j=0}^{k} C(j,k)\, T_{d_j\lambda}(\cdot,t), \tag{3.1.3}$$

where

$$C(j,k) = \begin{cases} \prod\limits_{\substack{i=0\\ i\neq j}}^{k} \dfrac{d_j}{d_j-d_i}, & k \neq 0 \\ 1, & k = 0. \end{cases} \tag{3.1.4}$$

In this chapter, for the operators $T_\lambda(\cdot,k,t)$, with $G \in T_\infty(\mathbb{R}^+)$, we obtain local direct and inverse theorems for a more general order $\varphi(\lambda^{-1/2})$ in L_p-norm $(1 \leq p < \infty)$ over contracting subintervals. All through this chapter, unless stated otherwise, the operators $T_\lambda(\cdot,t)$ corresponds to a kernel $G \in T_\infty(\mathbb{R}^+)$ and a given parameter $\alpha \in \mathbb{R}^+$.

A sectionwise summary is as follows : Section 3.2 contains basic results about the operators T_λ and $T_\lambda(\cdot,k,t)$ which will

be used in the later sections. Section 3.3 consists of local direct theorem for $T_\lambda(.,k,t)$ and sections 3.4 and 3.5 consist of local $O(\varphi)$ and $o(\varphi)$-inverse theorems for the operators $T_\lambda(.,k,t)$.

3.2 Basic Results.

Throughout this chapter we use the terminology that $I_j = [a_j,b_j]$, $j = 1,2$, where $0 < a_1 < a_2 < b_2 < b_1 < \infty$.

Kunwar [32] proved the following :

Lemma 3.2.1. If $G \in T(\mathbb{R}^+)$ and $G''(1)$ exists and is non-zero then

$$\lim_{\lambda\to\infty} \frac{\lambda^{1/2}\, a(\lambda)}{G^\lambda(1)} = \sqrt{\pi}\, \left[\frac{-2G(1)}{G''(1)}\right]^{1/2},$$

where $a(\lambda)$ is given by (3.1.2).

Immediately, we get the following

Corollary 3.2.2. If $G \in T(\mathbb{R}^+)$ and $G''(1) \neq 0$ then

$$\frac{a(\lambda)}{a^*(\lambda)} \to 1 \quad \text{as} \quad \lambda \to \infty,$$

where $a(\lambda)$ is given by (3.1.2) and

$$a^*(\lambda) = \int_0^\infty u^{\beta-2}\, G^\lambda(u)\, du \quad , \qquad \beta \in \mathbb{R}.$$

Before stating the next theorem we define the concept of a bounding function.

Definition 3.2.3. A continuous function $\Omega \geq 1$ defined on $\mathbb{R}^+$

is said to be bounding with respect to the operators T_λ, if for each compact subset K of $\mathbb{R}^+$, there exist positive numbers λ_K and M_K such that

$$T_{\lambda_K}(\Omega;t) < M_K \quad , \; t \in K$$

and, for any bounding function Ω, we define D_Ω to be the set of all measurable functions on $\mathbb{R}^+$ which are locally integrable on $\mathbb{R}^+$ such that $\limsup_{u \to 0} f(u)/\Omega(u)$, $\limsup_{u \to \infty} f(u)/\Omega(u)$ exist.

An asymptotic expansion for the moments of the operators T_λ is as follows :

<u>Theorem</u> 3.2.4. Let $G \in T(\mathbb{R}^+)$ and $k \in \mathbb{N}^o$. Then there exist constants $C_{k,r}$, $r \geq [\frac{k+1}{2}]$ such that the following asymptotic expansion is valid

$$t^{-k} T_\lambda((u-t)^k;t) = \sum_{r=[\frac{k+1}{2}]}^{\infty} \frac{C_{k,r}}{\lambda^r} \quad , \; (\lambda \to \infty).$$

<u>Lemma</u> 3.2.5. Let $G \in T(\mathbb{R}^+)$, $G''(1)$ exist and be non-zero and $k \in \mathbb{N}$. Then there exist λ_o and a constant $A_k > 0$ such that for all $\lambda > \lambda_o$ and $t \in \mathbb{R}^+$,

$$t^{-k} T_\lambda(|u-t|^k;t) \leq \frac{A_k}{\lambda^{k/2}} \, .$$

<u>Lemma</u> 3.2.6 : Let $G \in T(\mathbb{R}^+)$ and $m \in \mathbb{R}$. Then

$$\lim_{\lambda \to \infty} \frac{a(\lambda-m)}{a(\lambda)} = \{G(1)\}^{-m} \, ,$$

where $a(\lambda)$ is given by (3.1.2).

The following theorem enables us to extend our results to $G \in T_\infty(\mathbb{R}^+)$ by just proving them for $G \in T_\delta$ for an appropriate $\delta > 0$.

<u>Theorem</u> 3.2.7. Let $G, G^* \in T_\infty(\mathbb{R}^+)$ such that $G = G^*$ on $(1-\delta, 1+\delta)$ for some $\delta > 0$. Then, for any fixed $\ell > 0$ and $f \in L_p(\mathbb{R}^+)$ $(1 \leq p < \infty)$,

$$\| T_\lambda(f;t) - T_\lambda^*(f;t) \|_{L_p(0,\infty)} = O(\lambda^{-\ell}) \quad \|f\|_{L_p(\mathbb{R}^+)},$$

where T_λ, T_λ^* are the operators corresponding to the kernels G and G^*, respectively, the parameter α remaining the same.

<u>Proof</u>. Since the proof is similar to that of lemma 1.3.5, we only sketch it.

Let $\beta \in \mathbb{R}$ (we use it in the situations $\beta = \alpha$ and $\beta = \alpha+1$) and for any fixed number m, let

$$c = \int_0^\infty u^{\beta-2} \, G^m(u) \, du.$$

Indeed, if m is large enough then $c < \infty$.

Let $\sup\limits_{|u-1| > \delta} G(u) = \tau$; then $\tau < G(1)$. Hence, choose $\varepsilon > 0$ such that $\tau + \varepsilon < G(1)$. Since $G(u)$ is continuous at 1, there exists $\delta_0 > 0$ such that $\delta \geq \delta_0$ and

$$\inf_{|u-1| < \delta_0} G(u) > \tau + \varepsilon.$$

Let $\gamma = \min\{(1 \pm \delta_0)^{\alpha-2}, (1 \pm \delta_0)^{\beta-2}\}$.

Now,

$$a(\lambda) = \int_0^\infty u^{\alpha-2} G^\lambda(u)\, du \geq \int_{1-\delta_o}^{1+\delta_o} u^{\alpha-2} G^\lambda(u)\, du$$

$$\geq (\tau + \varepsilon)^\lambda \; 2\delta_o \gamma .$$

Hence,

$$\int_{|u-1|>\delta} u^{\beta-2} G^\lambda(u)\, du = \int_{|u-1|>\delta} u^{\beta-2} G^m(u)\, G^{\lambda-m}(u)\, du$$

$$\leq \tau^{\lambda-m} \int_0^\infty u^{\beta-2} G^m(u)\, du$$

$$\leq \tau^{\lambda-m}\, c.$$

Thus,

$$\frac{1}{a(\lambda)} \int_{|u-1|>\delta} u^{\beta-2} G^\lambda(u)\, du \leq \frac{\tau^{\lambda-m}\, c}{(\tau+\varepsilon)^\lambda\; 2\delta_o\, \gamma} = o(\lambda^{-\ell}) \quad .$$

We observe that

$$\frac{\int_{|u-1|>\delta} u^{\beta-2} G^\lambda(u)\, du}{\int_{1-\delta}^{1+\delta} u^{\alpha-2} G^\lambda(u)\, du} < \frac{c}{2\,\delta_o\, \tau^m \gamma} \left(\frac{\tau}{\tau+\varepsilon}\right)^\lambda = o(\lambda^{-\ell}) .$$

Hence, it is clear that

$$\frac{a(\lambda)}{a^*(\lambda)} = 1 + o(\lambda^{-\ell}) . \tag{3.2.2}$$

Now,

$$\| T_\lambda(f;t) - T_\lambda^*(f;t) \|_{L_p(\mathbb{R}^+)}$$

$$\leq \| \frac{1}{a(\lambda)} \int_{1-\delta}^{1+\delta} u^{\alpha-2} G^\lambda(u)\, f(t/u)\, du$$

(contd.)

$$- \frac{1}{a^*(\lambda)} \int_{1-\delta}^{1+\delta} u^{\alpha-2} G^{\lambda}(u) f(t/u) du||_{L_p(\mathbb{R}^+)}$$

$$+ ||\frac{1}{a(\lambda)} \int_{|u-1|>\delta} u^{\alpha-2} G^{\lambda}(u) f(t/u) du||_{L_p(\mathbb{R}^+)}$$

$$+ ||\frac{1}{a^*(\lambda)} \int_{|u-1|>\delta} u^{\alpha-2} G^{*\lambda}(u) f(t/u) du||_{L_p(\mathbb{R}^+)}$$

$$(3.2.3) \qquad = J_1 + J_2 + J_3, \text{ say.}$$

From (3.2.1), it easily follows that

$$(3.2.4) \qquad J_2 , J_3 = O(\lambda^{-\ell}) ||f||_{L_p(\mathbb{R}^+)} .$$

$$J_1 = ||(\frac{a(\lambda)}{a^*(\lambda)} - 1) \frac{1}{a(\lambda)} \int_{1-\delta}^{1+\delta} u^{\alpha-2} G^{\lambda}(u) f(t/u) du||_{L_p(\mathbb{R}^+)}$$

$$(3.2.5) \qquad = O(\lambda^{-\ell}) ||f||_{L_p(\mathbb{R}^+)} ,$$

from (3.2.1-2) and hence combining (3.2.3-5), we get the result.

<u>Lemma</u> 3.2.8. Let $f \in L_p(\mathbb{R}^+)$ $(1 \leq p < \infty)$ and $G \in T_\delta$ with $0 < \delta < \min (\frac{a_2-a_1}{a_1} , \frac{b_1-b_2}{b_1})$. Then

$$||T_\lambda(f;t)||_{L_p(I_2)} \leq (1+\delta)^{\frac{1}{p}} ||f||_{L_p(I_1)} .$$

<u>Proof</u>. We can easily see from the choice of δ that if $u \in (1-\delta, 1+\delta)$ and $t \in [a_2, b_2]$ then $t/u \in [a_1, b_1]$.

$$T_\lambda(f;t) = \frac{1}{a(\lambda)} \int_0^\infty u^{\alpha-2} G^{\lambda}(u) f(t/u) du.$$

Hence, by Jensen's inequality,

$$||T_\lambda(f;t)||^p \le \frac{1}{a(\lambda)} \int_{a_2}^{b_2} \int_0^\infty u^{\alpha-2}\, G^\lambda(u)\, |f(t/u)|^p \,du\,dt$$

$$= \frac{1}{a(\lambda)} \int_{a_2}^{b_2} \int_{1-\delta}^{1+\delta} u^{\alpha-2}\, G^\lambda(u)\, |f(t/u)|^p \,du\,dt$$

$$\le \frac{1}{a(\lambda)} \int_{a_1}^{b_1} \int_{1-\delta}^{1+\delta} u^{\alpha-1}\, G^\lambda(u)\, |f(t)|^p \,du\,dt$$

$$\le ||f||^p_{L_p(I_1)} (1+\delta),$$

and hence the lemma.

<u>Lemma</u> 3.2.9. Let $a < c < d < b$ and $X(u)$ be the characteristic function of the interval $[a,b]$. Further, let $f \in L_p(0,\infty)$ $(1 \le p < \infty)$ and $G \in T_\delta$ with $0 < \delta < \min\left(\frac{c-a}{a}, \frac{b-d}{b}\right)$. Then, for all $t \in [c,d]$,

$$T_\lambda((1-X)\, f;t) = 0.$$

The following lemma gives an expansion of the derivative of the operator.

<u>Lemma</u> 3.2.10. Let $G \in T_\delta$ and $\lambda > m \in \mathbb{N}$. Then there holds for $u,t \in \mathbb{R}^+$,

$$\frac{\partial^m}{\partial t^m}\{t^{\alpha-1}\, G^\lambda(t/u)\} = t^{\alpha-1}\, G^{\lambda-m}(t/u) \sum_{k=0}^{m} \sum_{\nu=0}^{\left[\frac{m-k}{2}\right]} \lambda^{\nu+k} \times$$

$$\times \{G'(t/u)\}^k\, g_{\nu,k,m}(t,u),$$

where the functions $g_{\nu,k,m}(t,u)$ are certain linear combination

of products of powers of u^{-1}, t^{-1} and $G^{(k)}(t/u)$, $k = 0,1,\ldots,m$ and are independent of λ.

We close this section by stating a result about linear combinations $T_\lambda(\cdot,k,t)$.

Theorem 3.2.11. Let $G \in T_\infty(\mathbb{R}^+)$, Ω be a bounding function for G and $f \in D_\Omega$. If at a point $t \in \mathbb{R}^+$, $f^{(2k+2)}$ exists, then there holds :

$$|T_\lambda(f;k,t) - f(t)| = O(\lambda^{-(k+1)}) ,$$

where $k = 0,1,2,\ldots$. Also, if $f^{(2k+2)}$ exists and is continuous on an open interval (c,d) containing $[a,b]$, then the result holds uniformly in $t \in [a,b]$.

3.3 Direct theorem.

In this section, we prove a direct theorem which contains both $O(\varphi)$ and $o(\varphi)$ direct theorems as corollaries.

Theorem 3.3.1. Let $f \in L_p(0,\infty)$ $(1 \le p < \infty)$. Then

$$||T_\lambda(f,k,t) - f(t)||_{L_p(I_2)}$$

$$\le M\{\omega_{2k+2}(f,\lambda^{-1/2},p,I_1) + \lambda^{-(k+1)} ||f||_{L_p(I)}\}.$$

Before going to the proof of the theorem, we shall prove the following results.

Lemma 3.3.2. Let $G \in T_\delta$ with $0 < \delta < \min\left(\frac{a_2-a_1}{a_1}, \frac{b_1-b_2}{b_1}\right)$ and $f \in L_p(0,\infty)$ $(1 \le p < \infty)$ have $2k+2$ derivatives on I_1 w

$f^{(2k+1)} \in A.C.(I_1)$ and $f^{(2k+2)} \in L_p(I_1)$. Then

$$\|T_\lambda(f,k,t) - f(t)\|_{L_p(I_2)}$$

$$\leq \frac{M}{\lambda^{k+1}} \{\|f^{(2k+2)}\|_{L_p(I_1)} + \|f\|_{L_p(I_1)}\}.$$

Proof. For $t \in I_2$ and $u \in I_1$, we have

$$(3.3.1) \quad f(u) = \sum_{i=1}^{2k+1} \frac{(u-t)^i}{i!} f^{(i)}(t)$$

$$+ \frac{1}{(2k+1)!} \int_t^u (u-w)^{2k+1} f^{(2k+2)}(w)\, dw.$$

Now, by the choice of δ, we have

$$(3.3.2) \quad T_\lambda(f;t) = \frac{1}{a(\lambda)} \int_0^\infty t^{\alpha-1} u^{-\alpha} \chi(u)\, G^\lambda(t/u)\, du,$$

where $\chi(u)$ is the characteristic function of I_1.

Thus, from (3.3.1-2), for $t \in I_2$, we get

$$(3.3.3) \quad T_\lambda(f,k,t)-f(t) = \sum_{i=1}^{2k+1} \frac{f^{(i)}(t)}{i!} T_\lambda((u-t)^i, k,t)$$

$$+ \frac{1}{(2k+1)!} T_\lambda(\int_t^u (u-w)^{2k+1} \chi(u)\, f^{(2k+2)}(w)\, dw, k,t).$$

Since $\sum_{j=0}^{k} c(j,k)\, d_j^{-m} = 0$, $m = 1,2,...,k$, using lemmas 3.2 and 0.6.3 , we get

$$(3.3.4) \quad \|\sum_{i=1}^{2k+1} \frac{f^{(i)}(t)}{i!} T_\lambda((u-t)^i,k,t)\|_{L_p(I_2)}$$

$$\leq \frac{M_1}{\lambda^{k+1}} \{\|f^{(2k+2)}\|_{L_p(I_2)} + \|f\|_{L_p(I_2)}\}.$$

Now, to estimate

$$Z = |T_\lambda(X(u) \int_t^u (u-w)^{2k+1} f^{(2k+2)}(w) \, dw; t)| ,$$

we consider the cases $p = 1$ and $p > 1$ seperately.

Suppose $p > 1$. Then

$$Z \leq T_\lambda(X(u) \ |u-t|^{2k+2} \cdot \frac{1}{|u-t|} \ |\int_t^u |f^{(2k+2)}(w)| \ dw| ; t)$$

$$= T_\lambda(|u-t|^{2k+2} \cdot H_f(u); t),$$

where $H_f(u)$ is the Hardy Littlewood majorant of $|f^{(2k+2)}|$ over the interval $[a_1, b_1]$. Hence, by Holder's inequality, we get

$$Z \leq \{T_\lambda(|u-t|^{(2k+2)q}; t)\}^{1/q} \{T_\lambda(X(u)|H_f(u)|^p; t)\}^{1/p} .$$

Thus, from lemmas 3.2.5 and 0.6.2 , we get

$$||Z||_{L_p(I_2)} \leq \frac{M_2}{\lambda^{k+1}} ||f^{(2k+2)}||_{L_p(I_1)} .$$

Hence,

$$(3.3.5) \quad ||T_\lambda(\int_t^u X(u)(u-w)^{2k+1} f^{(2k+2)}(w) \, dw, k, t)||_{L_p(I_2)}$$

$$\leq \frac{M_3}{\lambda^{k+1}} ||f^{(2k+2)}||_{L_p(I_1)} .$$

Suppose $p = 1$. Then, choose $r = r(\lambda)$ such that $r/\lambda^{1/2} \leq \max\{b_1-b_2, b_2-a_1\} \leq (r+1)/\lambda^{1/2}$. Then, for $t \in I_2$,

$$Z \leq \frac{1}{a(\lambda)} \int_{a_1}^{b_1} t^{\alpha-1} u^{-\alpha} G^{\lambda}(t/u) \, |u-t|^{2k+1} \times$$

$$\times \left| \int_t^u X(w) \, |f^{(2k+2)}(w)| \, dw \right| du$$

$$\leq \sum_{\ell=0}^{r} \Big\{ \frac{1}{a(\lambda)} \int_{t+\ell\lambda^{-1/2}}^{t+(\ell+1)\lambda^{-1/2}} t^{\alpha-1} u^{-\alpha} G^{\lambda}(t/u) \, |u-t|^{2k+1} \times$$

$$\times \Big(\int_t^{t+(\ell+1)\lambda^{-1/2}} X(w) \, |f^{(2k+2)}(w)| \, dw \Big) du$$

$$+ \frac{1}{a(\lambda)} \int_{t-(\ell+1)\lambda^{-1/2}}^{t-\ell\lambda^{-1/2}} t^{\alpha-1} u^{-\alpha} G^{\lambda}(t/u) \, |u-t|^{2k+1} \times$$

$$\times \Big(\int_{t-(\ell+1)\lambda^{-1/2}}^{t} X(w) \, |f^{(2k+2)}(w) dw \Big) du \Big\}.$$

Let $X_{t,c,d}$ denote the characteristic function of the interval $[t-c\lambda^{-1/2}, t+d\lambda^{-1/2}]$ where $c,d \in \mathbb{N}^0$. Then

$$Z \leq \sum_{\ell=1}^{r} \Big\{ \frac{1}{a(\lambda)} \int_{t+\ell\lambda^{-1/2}}^{t+(\ell+1)\lambda^{-1/2}} t^{\alpha-1} u^{-\alpha} G^{\lambda}(t/u) \, \lambda^2 \, \ell^{-4} \, |u-t|^{2k+5} \times$$

$$\times \Big(\int_t^{t+(\ell+1)\lambda^{-1/2}} X(w) \, X_{t,0,\ell+1}(w) \, |f^{(2k+2)}(w)| \, dw \Big) du$$

$$+ \frac{1}{a(\lambda)} \int_{t-(\ell+1)\lambda^{-1/2}}^{t-\ell\lambda^{-1/2}} t^{\alpha-1} u^{-\alpha} G^{\lambda}(t/u) \, \lambda^2 \, \ell^{-4} \, |u-t|^{2k+5} \times$$

$$\times \Big(\int_{t-(\ell+1)\lambda^{-1/2}}^{t} X(w) \, X_{t,\ell+1,0}(w) \, |f^{(2k+2)}(w)| \, dw \Big) du \Big\}$$

(contd.)

$$+ \frac{1}{a(\lambda)} \int_{a_2-\lambda^{-1/2}}^{b_2+\lambda^{-1/2}} t^{\alpha-1} u^{-\alpha} G^{\lambda}(t/u) \; |u-t|^{2k+1} \times$$

$$\times \int_{t-\lambda^{-1/2}}^{t+\lambda^{-1/2}} X(w) \, X_{\ell,1,1}(w) \; |f^{(2k+2)}(w)| \; dw\} \; du$$

$$\leq \sum_{\ell=1}^{r} \{ \frac{\lambda^2}{\ell^4} \cdot \frac{1}{a(\lambda)} \{ \int_{t+\ell\lambda^{-1/2}}^{t+(\ell+1)\lambda^{-1/2}} t^{\alpha-1} u^{-\alpha} G^{\lambda}(t/u) \; |u-t|^{2k+5} \times$$

$$\times \left(\int_{a_1}^{b_1} X_{t,0,\ell+1}(w) \; |f^{(2k+2)}(w)| \, dw \right) du$$

$$+ \int_{t-(\ell+1)\lambda^{-1/2}}^{t-\ell\lambda^{-1/2}} t^{\alpha-1} u^{-\alpha} G^{\lambda}(t/u) \; |u-t|^{2k+5} \times$$

$$\times \left(\int_{a_1}^{b_1} X_{t,\ell+1,0}(w) \; |f^{(2k+2)}(w)| \, dw \right) du \} \}$$

$$+ \frac{1}{a(\lambda)} \int_{a_2-\lambda^{-1/2}}^{b_2+\lambda^{-1/2}} t^{\alpha-1} u^{-\alpha} |u-t|^{2k+1} \; G^{\lambda}(t/u) \times$$

$$\times \int_{a_1}^{b_1} X_{t,1,1}(w) \; |f^{(2k+2)}(w)| \, dw) \; du \; .$$

Now, using lemma 3.2.5, we get

$$Z \leq M_5 \, \lambda^{-(2k+1)/2} \{ \sum_{\ell=1}^{r} \ell^{-4} \{ \int_{a_1}^{b_2} X_{t,0,\ell+1}(w) \; |f^{(2k+2)}(w)| \; dw$$

$$+ \int_{a_1}^{b_1} X_{t,\ell+1,0}(w) \; |f^{(2k+2)}(w)| \; dw\}$$

$$+ \int_{a_1}^{b_1} X_{t,1,1}(w) \; |f^{(2k+2)}(w)| \; dw \} .$$

Thus, using Fubini's theorem,

$$||Z||_{L_p(I_2)} \leq M_5 \lambda^{-(2k+1)/2} \sum_{\ell=1}^{r} \ell^{-4} \{ \int_{a_1}^{b_1} (\int_{a_1}^{b_1} \chi_{t,0,\ell+1}(w) \, dt) \times$$

$$\times |f^{(2k+2)}(w)| \, dw$$

$$+ \int_{a_1}^{b_1} (\int_{a_2}^{b_2} \chi_{t,\ell+1,0}(w) \, dt) \, |f^{(2k+2)}(w)| \, dw \}$$

$$+ \int_{a_1}^{b_1} (\int_{a_2}^{b_2} \chi_{t,1,1}(w) \, dt) \, |f^{(2k+2)}(w)| dw$$

$$= M_5 \lambda^{-(2k+1)/2} \{ \sum_{\ell=1}^{r} \ell^{-4} \{ \int_{a_1}^{b_1} |f^{(2k+2)}(w)| \; (\int_{w-(\ell+1)\lambda^{-1/2}}^{w} dt) \, dw$$

$$+ \int_{a_1}^{b_1} |f^{(2k+2)}(w)| \; (\int_{w}^{w+(\ell+1)\lambda^{-1/2}} dt) \, dw \}$$

$$+ \int_{a_1}^{b_1} |f^{(2k+2)}(w)| \; (\int_{w-\lambda^{1/2}}^{w+\lambda^{1/2}} dt) \, dw \}$$

$$\leq M_6 \lambda^{-(k+1)} \; ||f^{(2k+2)}||_{L_1(I_1)} ,$$

since $\sum_{\ell=1}^{\infty} \ell^{-3}$ is convergent.

Thus,

$$(3.3.6) \quad ||T_\lambda(\int_t^u (u-w)^{2k+1} \chi(u) \; f^{(2k+2)}(w) dw, k, t)||_{L_p(I_2)}$$

$$\leq M_6 \lambda^{-(k+1)} \; ||f^{(2k+2)}||_{L_1(I_1)} .$$

Hence, from (3.3.3-6), we get the lemma.

Proof of the theorem. It is sufficient to prove the theorem for all $G \in T_\delta$ with $0 < \delta < \min\left(\frac{a_2-a_1}{a_1}, \frac{b_1-b_2}{b_1}\right)$.

We have

$$\|T_\lambda(f,k,t) - f(t)\|_{L_p(I_2)}$$

$$\leq \|T_\lambda(f-f_{\eta,2k+2},k,t)\|_{L_p(I_2)}$$

$$+ \|T_\lambda(f_{\eta,2k+2},k,t) - f_{\eta,2k+2}(t)\|_{L_p(I_2)}$$

$$+ \|f_{\eta,2k+2}(t) - f(t)\|_{L_p(I_2)} .$$

Now, by lemma 3.2.9, we have

$$\|T_\lambda(f-f_{\eta,2k+2},k,t)\|_{L_p(I_2)}$$

$$\leq \|T_\lambda(\chi(f - f_{\eta,2k+2}),k,t)\|_{L_p(I_2)} ,$$

where χ is the characteristic function of I_1.

Thus, for sufficiently small $\eta > 0$, from lemmas 3.2.8 and 0.6.

$$(3.3.7) \quad \|T_\lambda(f-f_{\eta,2k+2},k,t)\|_{L_p(I_2)} \leq M_2\, \omega_{2k+2}(f,\eta,p,I_1)$$

and

$$(3.3.8) \quad \|f_{\eta,2k+2}(t) - f(t)\|_{L_p(I_2)} \leq M_3\, \omega_{2k+2}(f,\eta,p,I_1) .$$

Now, from lemma 3.3.2 and 0.6.5

$$(3.3.9)\quad \|T_\lambda(f_{\eta,2k+2},k,t) - f_{\eta,2k+2}(t)\|_{L_p(I_2)}$$

$$\leq \frac{M_3}{\lambda^{k+1}} \{\eta^{-(2k+2)}\, \omega_{2k+2}(f,\eta,p,I_1) + \|f\|_{L_p(I_1)}\}.$$

Thus, choosing $\eta = \lambda^{-1/2}$ and combining (3.3.7-9) we get the theorem.

Now, $\varphi \in \Phi_{2k+2}$ implies that $t^{2k+2}/\varphi(t) \to 0$ as $t \to 0$ and hence we get the following corollaries :

<u>Corollary</u> 3.3.3. Let $\varphi \in \Phi_{2k+2}$ and $f \in L_p(0,\infty)$ $(1 \leq p < \infty)$. Then

$$\omega_{2k+2}(f,t,p,I_1) = O(\varphi(t)) \qquad (t \to 0)$$

implies that

$$\|T_\lambda(f,k,t) - f(t)\|_{L_p(I_2)} = O(\varphi(\lambda^{-1/2})) \qquad (\lambda \to \infty).$$

<u>Corollary</u> 3.3.4. Let $\varphi \in \Phi_{2k+2}$ and $f \in L_p(0,\infty)$ $(1 \leq p < \infty)$. Then

$$\omega_{2k+2}(f,t,p,I_1) = o(\varphi(t)) \qquad (t \to 0)$$

implies that

$$\|T_\lambda(f,k,t) - f(t)\|_{L_p(I_2)} = o(\varphi(\lambda^{-1/2})) \qquad (\lambda \to \infty).$$

3.4 <u>$O(\varphi)$-inverse theorem.</u>

In this section we prove the corresponding $O(\varphi)$-inverse theorem of corollary 3.3.3.

<u>Theorem</u> 3.4.1. Let $f \in L_p(\mathbb{R}^+)$ $(1 \leq p < \infty)$ and $\varphi \in \Phi_{2k+2}$. The

$$(3.4.1) \qquad \|T_\lambda(f,k,t)-f(t)\|_{L_p(I_1)} = O(\varphi(\lambda^{-1/2})) \qquad (\lambda \to \infty)$$

implies that

$$(3.4.2) \qquad \omega_{2k+2}(f,t,p,I_2) = O(\varphi(t)) \qquad (t \to 0).$$

Before proceeding to the proof of the theorem, we shall prove the following results which will be useful in the proof of the theorem.

Lemma 3.4.2. Let $f \in L_p(\mathbb{R}^+)$ $(1 \leq p < \infty)$, $i,j \in \mathbb{N}^o$ and $G \in T_\delta$ with $0 < \delta < \min\left(\frac{a_2-a_1}{a_1}, \frac{b_1-b_2}{b_1}\right)$. Then

$$\|T_\lambda((u-t)^i \int_t^u (u-w)^j f(w)\,dw;t)\|_{L_p(I_2)} \leq M \lambda^{-\left(\frac{i+j+1}{2}\right)} \|f\|_{L_p(I_1)}$$

Proof. In view of the choice of δ, from lemma 3.2.9, we see that

$$\|T_\lambda((u-t)^i \int_t^u (u-w)^j f(w)\,dw;t)\|_{L_p(I_2)}$$

$$= \|T_\lambda((u-t)^i \chi(u) \int_t^u (u-w)^j f(w)\,dw;t)\|_{L_p(I_2)},$$

where χ is the characteristic function of I_1.

Now the proof follows exactly as in the case of the estimate of Z in the proof of lemma 3.3.2.

Lemma 3.4.3. Let $f \in Lp(\mathbb{R}^+)$ $(1 \leq p < \infty)$ and $G \in T_\delta$ with $0 < \delta < \min\left(\frac{a_2-a_1}{a_1}, \frac{b_1-b_2}{b_2}\right)$. Then

$$\|T_\lambda(|u-t|^i |f(u)|;t)\|_{L_p(I_2)} \leq M \lambda^{-i/2} \|f\|_{L_p(I_1)}.$$

Proof. Suppose $p > 1$. Then, by Holder's inequality,

$$T_\lambda(|u-t|^i |f(u)|;t)$$

$$\leq \{T_\lambda(|u-t|^{iq};t)\}^{1/q} \{T_\lambda(|f(u)|^p;t)\}^{1/q} \qquad (\frac{1}{p}+\frac{1}{q}=1).$$

Then, from lemmas 3.2.5 and 3.2.8, we get the result.

Suppose $p = 1$. Then

$$\int_{a_2}^{b_2} |T_\lambda(|u-t|^i |f(u)|;t)| \, dt$$

$$\leq \int_{a_2}^{b_2} \frac{1}{a(\lambda)} \int_0^\infty u^{\alpha-2} G^\lambda(u) |f(t/u)| \, |\frac{t}{u} - t|^i \, du \, dt$$

$$= \frac{1}{a(\lambda)} \int_{1-\delta}^{1+\delta} u^{\alpha-2} \, G^\lambda(u) \int_{a_2}^{b_2} |f(t/u)| \, |\frac{t}{u} - t|^i \, dt \, du$$

$$\leq \frac{M_1}{a(\lambda)} \int_{1-\delta}^{1+\delta} u^{\alpha-2} \, G^\lambda(u) \, |\frac{1}{u} - 1|^i \int_{a_2}^{b_2} |f(t/u)| \, dt \, du$$

$$\leq \frac{M_2}{a(\lambda)} \int_{1-\delta}^{1+\delta} u^{\alpha-2} \, G^\lambda(u) \, |\frac{1}{u} - 1|^i \int_{a_1}^{b_1} |f(t)| \, dt \, du$$

and hence, from lemma 3.2.5, we get that

$$||T_\lambda(|u-t|^i |f(u);t)||_{L_1(I_2)} \leq \frac{M_2}{\lambda^{i/2}} ||f||_{L_1(I_1)}$$

and hence the result is proved.

Lemma 3.4.4. Let $f \in L_p(0,\infty)$ $(1 \leq p < \infty)$ and $G \in T_\delta$ with $0 < \delta < \min(\frac{a_2-a_1}{a_1}, \frac{b_1-b_2}{b_1})$ and $\lambda > 2m \in \mathbb{N}$. Then

$$||T_\lambda^{(2m)}(f;t)||_{L_p(I_2)} = O(\lambda^m ||f||_{L_p(I_1)}).$$

If, in addition, f is $2m$-times differentiable on $\mathbb{R}^+$ such that $f^{(2m-1)}$ is absolutely continuous on $\mathbb{R}^+$ and $f^{(2m)} \varepsilon L_p(I_2)$. Then

$$||T_\lambda^{(2m)}(f;t)||_{L_p(I_2)} \leq M||f^{(2m)}||_{L_p(I_1)} .$$

Proof. We have

$$T_\lambda(f;t) = \frac{1}{a(\lambda)} \int_0^\infty t^{\alpha-1} u^{-\alpha} G^\lambda(t/u) f(u) du.$$

Hence, by lemma 3.2.10,

$$T_\lambda^{(2m)}(f;t) = \frac{1}{a(\lambda)} \int_0^\infty [t^{\alpha-1} G^\lambda(t/u)]^{(2m)} u^{-\alpha} f(u) du$$

$$= \frac{1}{a(\lambda)} \int_0^\infty u^{-\alpha} t^{\alpha-1} G^{\lambda-2m}(t/u) \sum_{k=0}^{2m} \sum_{\nu=0}^{[\frac{2m-k}{2}]} \lambda^{\nu+k}(G'(t/u))^k \times$$

$$\times g_{\nu,k,m}(t,u) f(u) du,$$

where the functions $g_{\nu,k,m}(t,u)$ are certain linear combinations of products of u^{-1}, t^{-1} and $G^{(k)}(t/u)$, $k = 0,\ldots,2m$ and are independent of λ.

Hence, using the boundedness of $g_{\nu,k,m}$ over the region of our interest,

$$\int_{a_2}^{b_2} |T_\lambda^{(2m)}(f;t)| dt$$

$$\leq M_1 \sum_{k=0}^{2m} \sum_{\nu=0}^{[\frac{2m-k}{2}]} \lambda^{\nu+k} \frac{1}{a(\lambda)} \int_{a_2}^{b_2} \int_0^\infty u^{-\alpha} t^{\alpha-1} G^{\lambda-2m}(t/u) \times$$

$$\times \{G'(t/u)\}^k |f(u)| du dt$$

$$= M_1 \sum_{k=0}^{2m} \sum_{\nu=0}^{[\frac{2m-k}{2}]} \lambda^{\nu+k} \frac{1}{a(\lambda)} \int_{a_2}^{b_2} \int_0^{\infty} y^{\alpha-2} G^{\lambda-2m}(y) |G'(y)|^k |f(t/y)| dy \, d$$

Now using the fact that $G \varepsilon T_\delta$

$$||T_\lambda^{(2m)}(f;t)||_{L_1(I_2)}$$

$$= M_2 \, ||f||_{L_1(I_1)} \sum_{k=0}^{2m} \sum_{\nu=0}^{[\frac{2m-k}{2}]} \lambda^{\nu+k} \times$$

$$\times \frac{1}{a(\lambda)} \int_0^{\infty} y^{\alpha-2} G^{\lambda-2m}(y) \, |G'(y)|^k \, dy$$

$$\leq M_2 ||f||_{L_1(I_1)} \sum_{k=0}^{2m} \sum_{\nu=0}^{[\frac{2m-k}{2}]} \lambda^{\nu+k} \frac{1}{a(\lambda)} \int_0^{\infty} y^{\alpha-2} G^{\lambda-2m}(y) \times$$

$$\times |y-1|^k \, |G''(\xi)|^k \, dy,$$

for some ξ lying between y and 1, since $G'(1) = 0$. Now, usin lemmas 3.2.5-6, we get

$$||T_\lambda^{(2m)}(f;t)||_{L_1(I_2)} \leq M_3 ||f||_{L_1(I_1)} \sum_{k=0}^{2m} \sum_{\nu=0}^{[\frac{2m-k}{2}]} \lambda^{\nu+k/2}$$

$$\leq M_4 \lambda^m \, ||f||_{L_1(I_1)}.$$

Similarly, we can prove

$$||T_\lambda^{(2m)}(f;t)||_{L_\infty(I_2)} \leq M_5 \lambda^m \, ||f||_{L_\infty(I_1)}$$

and hence we get the result from Riesz-Thorin's interpolatior theorem.

Since $f^{(2m-1)}$ is absolutely continuous, $f^{(2m)} \in L_p(\mathbb{R}^+)$ and $G \in T_\delta$, we can write

$$T_\lambda^{(2m)}(f;t) = T_\lambda(f^{(2m)},t)$$

and hence from lemma 3.2.8, the second part of the result follows.

Proof of theorem 3.4.1. In view of lemma 3.2.7, it is sufficient to prove the theorem for all $G \in T_\delta$ with $0 < \delta < \min\left(\frac{a_2-a_1}{a_1}, \frac{b_1-b_2}{b_1}\right)$. Choose x_i, y_i, $i = 1,2,3,4$ such that $a_1 < x_i < x_{i+1} < a_2 < b_2 < y_{i+1} < y_i < b_1$. Then

$$||\Delta_\gamma^{2k+2} f(t)||_{L_p[x_3,y_3]} \leq ||\Delta_\gamma^{2k+2}\{f(t)-T_\lambda(f,k,t)\}||_{L_p[x_3,y_3]}$$

$$+ ||\Delta_\gamma^{2k+2} T_\lambda(f,k,t)||_{L_p[x_3,y_3]}.$$

By lemma 0.6.1

$$(3.4.3) \quad ||\Delta_\gamma^{2k+2} T_\lambda(f,k,t)||_{L_p[x_3,y_3]}$$

$$= ||\int_0^\gamma \cdots \int_0^\gamma T_\lambda^{(2k+2)}\left(f,k,t+\sum_{i=1}^{2k+2} z_i\right) dz_1 \cdots dz_{2k+2}||_{L_p[x_3,y_3]}.$$

Now, by repeated applications of Jensen's inequality and Fubini's theorem, for all γ sufficiently small, we have

$$\int_{x_3}^{y_3} |\int_0^\gamma \cdots \int_0^\gamma T_\lambda^{(2k+2)}\left(f,k,t+\sum_{i=1}^{2k+2} z_i\right) dz_1 \cdots dz_{2k+2}|^p\, dt$$

(contd.)

$$\leq \gamma^{(2k+2)(p-1)} \int_0^\gamma \cdots \int_0^\gamma ||T_\lambda^{(2k+2)}(f,k,t)||^p_{L_p[x_3,y_3]} \, dz_1 \cdots dz_{2k+2}$$

$$= \gamma^{(2k+2)p} \, ||T_\lambda^{(2k+2)}(f,k,t)||^p_{L_p[x_3,y_3]}.$$

Hence, from (3.4.3),

$$||\Delta_\gamma^{2k+2} \, T_\lambda(f,k,t)||_{L_p[x_3,y_3]}$$

$$\leq \gamma^{(2k+2)} \, ||T_\lambda^{(2k+2)}(f,k,t)||_{L_p[x_3,y_3]}.$$

Suppose supp $f \subset (x_3,y_3)$. Then, from lemma 3.4.4

$$||\Delta_\gamma^{2k+2} \, T_\lambda(f,k,t)||_{L_p[x_3,y_3]}$$

$$\leq \gamma^{(2k+2)} \, ||T_\lambda^{(2k+2)}(f-f_{\eta,2k+2},k,t)||_{L_p[x_2,y_2]}$$

$$+ \, ||T_\lambda^{(2k+2)}(f_{\eta,2k+2},k,t)||_{L_p[x_2,y_2]} \, ,$$

which, for all η sufficiently small, is

$$\leq M_1 \, \gamma^{2k+2} \, \{\lambda^{k+1} \, ||f-f_{\eta,2k+2}||_{L_p[x_2,y_2]}$$

$$+ \, ||f_{\eta,2k+2}^{(2k+2)}||_{L_p[x_2,y_2]}\}.$$

Now from lemma 0.6.5 and the fact that supp $f \subset (x_3,y_3)$, we get

$$(3.4.4) \quad ||\Delta_\gamma^{2k+2} \, T_\lambda(f,k,t)||_{L_p[x_3,y_3]}$$

$$\leq M_2 \, \gamma^{2k+2} \, (\lambda^{k+1} + \frac{1}{\eta^{2k+2}}) \, \omega_{2k+2}(f,\eta,p,[x_3,y_3]).$$

Also, since supp $f \subset (x_3, y_3)$ and γ is sufficiently small, from hypothesis (3.4.1), we get

(3.4.5) $\|\Delta_\gamma^{2k+2}\{f(t)-T_\lambda(f,k,t)\}\|_{L_p[x_3,y_3]} = O(\varphi(\lambda^{-1/2}))\ (\lambda \to \infty).$

Thus choosing $\lambda = \eta^{-2}$, from (3.4.4-5), we get

$$\|\Delta_\gamma^{2k+2} f(t)\|_{L_p[x_3,y_3]} \leq \{\varphi(\eta) + \gamma^{2k+2}(n^{k+1} + \frac{1}{\eta^{2k+2}}) \times$$

$$\times\ \omega_{2k+2}(f,\eta,p,[x_3,y_3])\},$$

which implies that

$$\omega_{2k+2}(f,t,p,[x_3,y_3]) \leq M_4\ \{\varphi(\eta) + (\frac{t}{\eta})^{2k+2} \times$$

$$\times\ \omega_{2k+2}(f,\eta,p,[x_3,y_3])\}.$$

The conclusion, now, follows from lemma 1.2.2.

Next, we prove the theorem for general f.

Let $g \in C_0^{2k+2}$ be such that supp $g \subset (x_3, y_3)$ and $g \equiv 1$ on $[x_4, y_4]$. Then, it is sufficient to prove that

(3.4.6) $\|T_\lambda(fg,k,t) - (fg)(t)\|_{L_p[x_3,y_3]} = O(\varphi(\lambda^{-1/2}))\ (\lambda \to \infty).$

We prove this by induction as follows : First we prove the result for all $\varphi \in \Phi_1$. Later, assuming the theorem for all $\varphi \in \Phi_r$, for some r such that $1 \leq r \leq 2k+1$, we prove it for all $\varphi \in \Phi_{r+1}$.

Let $\varphi \in \Phi_1$. Then

$$(3.4.7) \quad \|T_\lambda(fg,k,t) - (fg)(t)\|_{L_p[x_3,y_3]}$$

$$\leq \|T_\lambda((f(u)-f(t))\,g(t),k,t)\|_{L_p[x_3,y_3]}$$

$$+ \|T_\lambda(f(u)(g(u)-g(t)),k,t)\|_{L_p[x_3,y_3]} \, .$$

Thus, using the hypothesis (3.4.1) and lemma 3.4.3 for the first and second terms on the right hand side of (3.4.7), we get (3.4.6), since $t = o(\varphi(t))$ $(t \to 0)$. Hence the theorem is proved for all $\varphi \in \Phi_1$.

Assume that for some r such that $1 \leq r \leq 2k+1$, the result is true for all $\varphi \in \Phi_r$. Let $\varphi \in \Phi_{r+1}$. Then,

$$\varphi^*(t) = \frac{\varphi(t)}{t} \in \Phi_r \, .$$

Hence, by induction hypothesis, we get

$$\omega_{2k+2}(f,t,p,[x_1,y_1]) = O(\varphi^*(t)) \qquad (t \to 0)$$

which, by lemma 1.2.4, implies that

$$\omega_{2k+1}(f,t,p,[x_1,y_1]) = O(\varphi^*(t)) \qquad (t \to 0) .$$

Hence, for $j = 2k+1,\ 2k+2$

$$(3.4.8) \quad \lambda^{-\frac{1}{2}}\,\omega_j(f,\lambda^{-1/2},p,[x_1,y_1]) = O(\varphi(\lambda^{-1/2})) \qquad (\lambda \to \infty) .$$

Now, in view of (3.4.7), it is sufficient to prove that

$$(3.4.9) \quad J = \|T_\lambda(f(u)(g(u)-g(t)),k,t)\|_{L_p[x_3,y_3]} = O(\varphi(\lambda^{-1/2}))$$
$$(\lambda \to \infty) .$$

We have

(3.4.10) $\quad J \leq J_1 + J_2 + J_3$, where

$$J_1 = \|T_\lambda((f(u)-f_{\eta,2k+2}(u))\,(g(u) - g(t)),k,t)\|_{L_p[x_3,y_3]},$$

$$J_2 = \|T_\lambda((f_{\eta,2k+2}(u)-f_{\eta,2k+2}(t))\,(g(u)-g(t)),k,t)\|_{L_p[x_3,y_3]}$$

and

$$J_3 = \|T_\lambda(f_{\eta,2k+2}(t)\,(g(u)-g(t)),k,t)\|_{L_p[x_3,y_3]}.$$

For some ξ lying between u and t,

$$J_1 = \|T_\lambda((f(u)-f_{\eta,2k+2}(u))\,(u-t)\,g'(\xi),k,t)\|_{L_p[x_3,y_3]}.$$

Now, choosing $\lambda = \eta^{-2}$, from lemmas 3.4.3 and 0.6.5 , we see that

(3.4.11) $\quad J_1 \leq \lambda^{-1/2}\,\omega_{2k+2}(f,\lambda^{-1/2},p,[x_1,y_1])$,

which, in view of (3.4.8), implies that

(3.4.12) $\quad J_1 = O(\varphi(\lambda^{-1/2})) \qquad (\lambda \to \infty).$

Also by theorem 3.2.11 and lemma 0.6.5

(3.4.13) $\quad J_3 \leq \dfrac{M_5}{\lambda^{k+1}}$.

Since $\varphi \in \Phi_{r+1} \subset \Phi_{2k+2}$, $t^{2k+2}/\varphi(t) \to 0$ as $t \to 0$. Hence,

(3.4.14) $\quad J_3 = O(\varphi(\lambda^{-1/2})) \qquad (\lambda \to \infty).$

Again for some ξ lying between u and t,

$$(f_{\eta,2k+2}(u) - f_{\eta,2k+2}(t))\ (g(u) - g(t))$$

$$= \left\{ \sum_{i=1}^{2k+1} \frac{(u-t)^i}{i!} f_{\eta,2k+2}^{(i)}(t) + \frac{1}{(2k+1)!} \int_t^u (u-w)^{2k+1} f_{\eta,2k+2}^{(2k+2)}(w)\,dw \right\} \times$$

$$\times \left\{ \sum_{i=1}^{2k} \frac{(u-t)^i}{i!} g^{(i)}(t) + \frac{(u-t)^{2k+1}}{(2k+1)!} g^{(2k+1)}(\xi) \right\}.$$

Therefore,

$$J_2 \leq \frac{1}{(2k+1)!} \sum_{i=1}^{2k} \left\| \frac{g^{(i)}(t)}{i!} T_\lambda((u-t)^i \times \right.$$

$$\left. \times \int_t^u (u-w)^{2k+1} f_{\eta,2k+2}^{(2k+2)}(w)\,dw, k, t) \right\|_{L_p[x_3, y_3]}$$

$$+ \frac{1}{((2k+1)!)^2} \left\| T_\lambda(g^{(2k+1)}(\xi)(u-t)^{2k+1} \times \right.$$

$$\left. \times \int_t^u (u-w)^{2k+1} f_{\eta,2k+2}^{(2k+2)}(w)\,dw, k, t) \right\|_{L_p[x_3, y_3]}$$

$$+ \left\{ \sum_{i=1}^{2k+1} \sum_{j=1}^{2k} \frac{1}{i!j!} \left\| f_{\eta,2k+2}^{(i)}(t)\ g^{(j)}(t) T_\lambda((u-t)^{i+j}, k, t) \right\|_{L_p[x_3, y_3]} \right.$$

$$+ \frac{1}{(2k+1)!} \left\{ \sum_{j=1}^{2k+1} \frac{1}{i!} \right\| f_{\eta,2k+2}^{(i)}(t) \times$$

$$\times T_\lambda((u-t)^{2k+1+i}\ g^{(2k+1)}(\xi), k, t) \|_{L_p[x_3, y_3]}$$

$$(3.4.15) \qquad = R_1 + R_2 + R_3 + R_4, \text{ say.}$$

By lemma 3.4.2,

$$R_1 \leq M_6 \left\{ \sum_{i=1}^{2k} \lambda^{-(k+1+i/2)} \| f_{\eta,2k+2}^{(2k+2)} \|_{L_p[x_3, y_3]} \right\}$$

and

$$R_2 \leq M_7 \{\lambda^{-(2k+3/2)} \;||f^{(2k+2)}_{\eta,2k+2}||_{L_p[x_2,y_2]}\}.$$

By lemma 0.6.5 and choosing $\lambda = \eta^{-2}$ we get

$$(3.4.16) \quad R_1,\ R_2 \leq M_8 \;\; \lambda^{-1/2}\, \omega_{2k+2}\;(f,\lambda^{-1/2},p,\ [x_1,y_1])$$

which, by (3.4.8) implies that

$$(3.4.17) \quad R_1,R_2 = O(\varphi(\lambda^{-1/2})) \qquad (\lambda \to \infty).$$

It follows from theorem 3.2.4 and the fact that $\sum_{j=0}^{k} C(j,k)\ d_j^{-m} = 0,\ m = 1,2,\ldots,k,$ that

$$R_{\smile} \leq \frac{M_9}{\lambda^{k+1}} \left(\sum_{i=1}^{2k+1} \;||f^{(i)}_{\eta,2k+2}||_{L_p[x_3,y_3]}\right).$$

Clearly,

$$R_4 \leq \frac{M_{10}}{\lambda^{k+1}} \left(\sum_{i=1}^{2k+1} \;||f^{(i)}_{\eta,2k+2}||_{L_p[x_3,y_3]}\right).$$

Now using lemma 0.6.3 , we get

$$R_3,R_4 \leq \frac{M_{11}}{\lambda^{k+1}} \{||\,f^{(2k+1)}_{\eta,2k+2}||_{L_p[x_3,y_3]} + ||f_{\eta,2k+2}||_{L_p[x_3,y_3]}\}$$

which, by lemma 0.6.5 , after taking $\lambda = \eta^{-2}$, implies that

$$(3.4.18) \quad R_3,R_4 \leq \lambda^{-1/2}\ \omega_{2k+1}(f,\lambda^{-1/2},p,[x_1,y_1])\lambda^{-(k+1)}\;||f||_{L_p(}$$

Now, from (3.4.8) and the fact that $\dfrac{\lambda^{-(k+1)}}{\varphi(\lambda^{-1/2})} \to 0$ as $\lambda \to \infty$ we get that

$$(3.4.19) \quad R_3,R_4 = O(\varphi(\lambda^{-1/2}))$$

Hence, from (3.4.15), (3.4.17) and (3.4.19),

$$J_2 = O(\varphi(\lambda^{-1/2})) \qquad (\lambda \to \infty).$$

Thus, $J = J_1 + J_2 + J_3 = O(\varphi(\lambda^{-1/2}))$ $\qquad (\lambda \to \infty)$

and hence the result follows.

Corollary 3.4.5. Let $f \in L_p(\mathbb{R}^+)$ and for some $r \in \mathbb{N}$ such that $1 \leq r \leq 2k+2$, $\varphi \in \Phi_r$. Then

$$\|T_\lambda(f,k,t) - f(t)\|_{L_p(I_1)} = O(\varphi(\lambda^{-1/2})) \qquad (\lambda \to \infty)$$

implies that

$$\omega_r(f,t,p,I_2) = O(\varphi(\lambda^{-1/2})).$$

Proof. Recursive use of lemma 1.2.4 with theorem 3.4.1.

3.5 o(φ)-inverse theorem.

In this section we prove the inverse theorem corresponding to corollary 3.3.4.

Theorem 3.5.1. Let $f \in L_p(\mathbb{R}^+)$ $(1 \leq p < \infty)$ and $\varphi \in \Phi_{2k+2}$. Then

$$(3.5.1) \qquad \|T_\lambda(f,k,t) - f(t)\|_{L_p(I_1)} = o(\varphi(\lambda^{-1/2})) \qquad (\lambda \to \infty)$$

implies that

$$(3.5.2) \qquad \omega_{2k+2}(f,t,p,I_2) = o(\varphi(t)) \qquad (t \to 0).$$

Proof. Choose x_i, y_i, $i = 1,2,3,4$ such that $a_1 < x_i < x_{i+1} < a_2 < b_2 < y_{i+1} < y_i < b_1$.

By hypothesis (3.5.1), if γ is sufficiently small and supp $f \subset (x_3, y_3)$ then

(3.5.3) $$||\Delta_\gamma^{2k+2}\{f(t)-T_\lambda(f,k,t)\}||_{L_p[x_3,y_3]} = o(\varphi(\lambda^{-1/2})) \quad (\lambda \to \infty).$$

Now, defining

$$\psi(x) = ||\Delta_\gamma^{2k+2}\{f(t)-T_\lambda(f,k,t)\}||_{L_p[x_3,y_3]},$$

where $x = \lambda^{-1/2}$, we observe that

$$\psi(t) = o(\varphi(t)) \qquad (t \to 0).$$

Also, from (3.4.4) of theorem 3.4.1, we have

(3.5.4) $$||\Delta_\gamma^{2k+2} T_\lambda(f,k,t)||_{L_p[x_3,y_3]}$$

$$\leq M_1 \gamma^{2k+2} (\lambda^{k+1} + \frac{1}{\eta^{2k+2}}) \omega_{2k+2}(f,\eta,p,[x_3,y_3]).$$

Hence, combining (3.5.3-4) and choosing $\lambda = \eta^{-2}$, we get

$$||\Delta_\gamma^{2k+2} f(t)||_{L_p[x_3,y_3]} = M_2 \{\psi(\eta) + \gamma^{2k+2} (\lambda^{k+1} + \frac{1}{\eta^{2k+2}}) \times$$
$$\times \omega_{2k+2}(f,t,p,[x_3,y_3])\},$$

which, by lemma 1.2.3, after taking supremum over all $\gamma \leq t$, implies the result.

Thus we have proved the theorem for all f such that supp $f \subset (x_3, y_3)$.

Hence, to prove the theorem for general f, it is sufficient to prove that

(3.5.5) $\quad ||(fg)(t) - T_\lambda(fg,k,t)||_{L_p[x_3,y_3]} = o(\varphi(\lambda^{-1/2})) \quad (\lambda \to 0)$,

where $g \in C_0^{2k+2}$ such that supp $g \subset (x_3,y_3)$ and $g \equiv 1$ on $[x_4,y_4]$. Now, if $\varphi \in \Phi_1$, we have

$$||T_\lambda(fg,k,t) - (fg)(t)||_{L_p[x_3,y_3]}$$

$$\leq ||T_\lambda((f(u)-f(t))\, g(t),k,t)||_{L_p[x_3,y_3]}$$

$$+ ||T_\lambda(f(u)(g(u)-g(t)),k,t)||_{L_p[x_3,y_3]},$$

$$= o(\varphi(\lambda^{-1/2})),$$

by (3.5.1) and lemma 3.4.3. Hence the theorem is proved for all $\varphi \in \Phi_1$.

Next, assume the result for all $\varphi \in \Phi_r$, for some r such that $1 \leq r \leq 2k+1$, and let $\varphi \in \Phi_{r+1}$. Let (3.5.1) hold. Then, since $\varphi^*(t) = \dfrac{\varphi(t)}{t} \in \Phi_r$, by our assumption, we get

(3.5.6) $\quad \omega_{2k+2}(f,t,p,[x_1,y_1]) = o(\varphi^*(t)) \qquad (t \to 0)$

which, by lemma 1.2.5, implies that

(3.5.7) $\quad \omega_{2k+1}(f,t,p,[x_1,y_1]) = o(\varphi^*(t)) \qquad (t \to \infty)$.

Clearly, by hypothesis (3.5.1), we have

$$||T_\lambda((f(u)-f(t))g(t),k,t)||_{L_p[x_3,y_3]} = o(\varphi(\lambda^{-1/2})) \quad (\lambda \to \infty).$$

Hence,

$$J = \|T_\lambda(fg,k,t) - (fg)(t)\|_{L_p[x_3,y_3]}$$

$$\leq \|T_\lambda((f(u)-f(t))g(t),k,t)\|_{L_p[x_3,y_3]}$$

$$+ \|T_\lambda((f(u)-f_{\eta,2k+2}(u))(g(u)-g(t)),k,t)\|_{L_p[x_3,y_3]}$$

$$+ \|T_\lambda((f_{\eta,2k+2}(u)-f_{\eta,2k+2}(t))(g(u)-g(t)),k,t)\|_{L_p[x_3,y_3]}$$

$$+ \|T_\lambda(f_{\eta,2k+2}(t)(g(u)-g(t)),k,t)\|_{L_p[x_3,y_3]}$$

(3.5.8) $\qquad = o(\varphi(\lambda^{-1/2})) + J_1 + J_2 + J_3$, say.

For some ξ lying between u and t,

$$J_1 \leq \|T_\lambda((f(u)-f_{\eta,2k+2}(u))(u-t)g'(\xi),k,t)\|_{L_p[x_3,y_3]},$$

which after choosing $\lambda = \eta^{-2}$, from lemmas 3.4.3 and 0.6.5, implies that

$$J_1 \leq \lambda^{-1/2}\,\omega_{2k+1}(f,\lambda^{-1/2},p,[x_1,y_1]).$$

Applying, theorem 3.2.11 and lemma 0.6.5, we get

$$J_3 \leq \frac{M_3}{\lambda^{k+1}}.$$

Hence, using (3.5.6) and the fact that $t^{2k+2}/\varphi(t) \to 0$ as $t \to 0$ respectively, for J_1, J_3, we get

(3.5.9) $\qquad J_1, J_3 = o(\varphi(\lambda^{-1/2})) \qquad (\lambda \to \infty).$

The quantity J_2 has already been considered and estim[ated] in (3.4.15-17) of the proof of theorem 3.4.1 as

$J_2 \le R_1+R_2+R_3+R_4$, where

$$R_1, R_2 \le \lambda^{-1/2}\omega_{2k+2}(f,\lambda^{-1/2},p,[x_1,y_1])$$

and

$$R_3, R_4 \le \lambda^{-1/2}\omega_{2k+1}(f,\lambda^{-1/2},p,[x_1,y_1]) + \lambda^{-(k+1)} \; ||f||_{L_p(I_1)} .$$

Hence, from (3.5.6-7) and the fact that $t^{2k+2}/\varphi(t) \to 0$ as $t \to 0$, we get, for $i = 1,2,3,4$

$$R_i = o(\varphi(\lambda^{-1/2})) \qquad (\lambda \to \infty).$$

Thus $J_2 = o(\varphi(\lambda^{-1/2}))$ $(\lambda \to \infty)$ and hence the result follows from (3.5.8-9).

An immediate consequence of this theorem and lemma 1.2.5 is the following

<u>Corollary</u> 3.5.2. Let $f \in L_p(\mathbb{R}^+)$ and for some $r \in \mathbb{N}$, such that $1 \le r \le 2k+2$, $\varphi \in \Phi_r$. Then

$$||T_\lambda(f,k,t - f(t)||_{L_p(I_1)} = o(\varphi(\lambda^{-1/2}) \qquad (\lambda \to \infty)$$

implies that

$$\omega_r(f,t,p,I_2) = o(\varphi(t)) \qquad (t \to 0).$$

CHAPTER IV

φ - INVERSE THEOREMS FOR LINEAR COMBINATIONS AND INTERPOLATORY MODIFICATIONS OF REGULAR EXPONENTIAL TYPE OPERATORS

4.1 Introduction.

In this chapter, we study the linear combinations $S_n(\cdot,k,t)$ and interpolatory modifications $S_{n,m}(\cdot,t)$ of the regular exponential type operators $S_n(\cdot,t)$. These regular exponential type operators were, first, introduced by May [39], for approximation of functions belonging to the class $C(A,B)$ of bounded, continuous functions on (A,B) $(-\infty \le A < B \le \infty)$ into $C^{\infty}(A,B)$, having the form

$$S_n(f,t) = \int_A^B W(n,t,u)\ f(u)du,$$

where $S_n(1,t) = 1$, $t \in (A,B)$, $f \in C(A,B)$, $W(n,t,u) \ge 0$ and satisfies the following:

1. $$\frac{\partial}{\partial t} W(n,t,u) = \frac{n}{p(t)} W(n,t,u)(u-t), u,t \in (A,B),$$

where $p(t)$ is a polynomial of degree ≤ 2 and $p(t) > 0$ for $t \in (A,B)$.

2. For $r \in \mathbb{N}$,

$$\frac{d^r}{dt^r} S_n(f;t) = \int_A^B \{\frac{\partial^r}{\partial t^r} W(n,t,u)\} f(u)du .$$

The integrals in above are in the Lebesgue-Stieltjes sense for which $W(n,t,u)$ is the distributional kernel.

3. The kernel $W(n,t,u)$ (regarded as a function in t and u) is measurable on $(A,B) \times (A,B)$ and

$$\int_A^B W(n,t,u)\,dt = a(n) \;, \quad u \in (A,B)$$

where $a(n)$ is a rational function of n with $a(n) \to 1$ as $n \to \infty$ and for each fixed $u \in (A,B)$, $m \in \mathbb{N}^0$ and n sufficiently large

$$t^m p(t) W(n,t,u) \to 0 \quad \text{as} \quad t \to A \text{ or } B.$$

The class of operators which satisfy (1) and (2) above are called exponential type operators. In addition, if they satisfy (3) also, they are called regular. It has recently been shown by Sinha [61] that for $f \in L_p[A,B]$ $(1 \leq p < \infty)$, regular exponential type operators constitute an approximation method. Well known examples of regular exponential type operators are the operators of Post-Widder and Gauss-Weierstrass, defined respectively by

$$S_n(f;t) = \frac{1}{\Gamma(n)} \left(\frac{n}{t}\right)^n \int_0^\infty e^{-nu/t} u^{n-1} f(u)\,du$$

and

$$S_n(f;t) = \left(\frac{n}{2\pi}\right)^{1/2} \int_{-\infty}^\infty e^{-n(u-t)^2/2} f(u)\,du.$$

Examples of exponential type operators which are not regular are the Bernstein polynomials, the Szász operators and the Baskakov operators, defined respectively by

$$S_n(f;t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} f(\tfrac{k}{n}) ,$$

$$S_n(f;t) = e^{-nt} \sum_{k=0}^{\infty} \frac{1}{k!} (nt)^k f(\tfrac{k}{n})$$

and

$$S_n(f;t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \theta_n^{(k)}(t)\, t^k f(\tfrac{k}{n}),$$

where the family $\{\theta_n\}$ satisfies certain appropriate conditions.

Also, the operators $L_n(f;t)$ of Rathore [48] defined by

$$L_n(f;t) = (1+t)^{-(n+1)} \sum_{s=0}^{\infty} \binom{n+s}{s} \left(\frac{t}{1+t}\right)^s f\left(\frac{s}{n+p}\right)$$

give rise to the operators

$$S_n(f;t) = (1+t)^{-n} \sum_{s=0}^{\infty} \binom{n+s-1}{s} \left(\frac{t}{1+t}\right)^s f(\tfrac{s}{n}),$$

where $S_n \equiv L_{n-1}$ with $p = 1$, which are of exponential type but are not regular.

All through this chapter it will be assumed that $\{S_n\}$ are regular exponential type operators and that $f \in L_p[A,B]$.

Let $d_o, d_1, \ldots, d_k$ be $k+1$ distinct positive integers. Then the linear combination $S_n(f;k,t)$ is defined by

$$S_n(f,k,t) = \sum_{j=0}^{k} C(j,k)\, S_{d_j n}(f,t) ,$$

where

$$C(j,k) = \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{d_j}{d_j-d_i}, \quad k \neq 0.$$

and

$$C(0,0) = 1 .$$

The interpolatory modification $S_{n,m}(\cdot,t)$, introduced by Sinha [61], of the regular exponential type operators $S_n(\cdot,t)$, is a modification of $S_n(\cdot,t)$, slightly different from the case of $P_{n,m}(\cdot,t)$, obtained by replacing function value $f(u)$ at the point 'u' by Newton's interpolation polynomial of $(m+1)$th degree based at the nodes $u,\ u+n^{-1/2},\ldots,u+mn^{-1/2}$ and devided by $(|u-t|^{m_o+2}+1)$, $m+1 \leq m_o$. The latter operators, thus, become L_p-bounded (as discussed by Sinha [61]).

Thus, for $f \in L_p[A,B]$, interpolatory modification, $S_{n,m}(f;t)$, of order m, of regular exponential type operators $S_n(f;t)$ is defined as follows:

$$S_{n,m}(f;t) = \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} \times$$

$$\times \left\{ \sum_{j=0}^{m} \frac{n^{j/2}}{j!} \left(\prod_{i=0}^{j-1} \left(t-u-\frac{i}{n^{1/2}}\right)\right) \Delta^j f(u) \right\} du,$$

where $m \leq m_o$ and Δ denotes $\Delta_{n^{-1/2}}$.

The class of exponential operators was introduced by May [39] . Giving a unified treatment for this class, May [39] obtained sup-norm local inverse theorems for Butzer type linear combinations [14] of members of this class. With the additional assumption of regularity he also obtained saturation theorems for these linear combinations This treatment is based on intermediate space approach involving a Peetre's K-functional for inverse theorems and is distribution theory oriented for saturation theorems.

Later, Agrawal [1] studied the inverse and saturation theorems in simultaneous approximation for exponential type operators. Infact , he considered the problem in a more general frame work of two types of iterative combinations which contain the iterative combinations of Micchelli as well as the ordinary linear combinations considered by May [39] as special cases.

As noted before, Sinha [61] proved that regular exponential type operators are L_p-approximation methods. He, then, obtained local direct, inverse and saturation theorems with respect to L_p-norm ($1 \leq p < \infty$) for the linear combinations and interpolatory modifications of regular exponential type operators. He proved inverse theorems for the order $O(n^{-\alpha/2})$ ($0 < \alpha < 2k+2$).

In this chapter, we extend the results of Sinha [61] to a more general order $\varphi(n^{-1/2})$, which generalise the

inverse theorems for linear combinations as well as interpolatory modifications of regular exponential type operators in L_p-norm $(1 \leq p < \infty)$. These are also local in nature over contracting sub-intervals.

The layout of this chapter is as follows: Section 4.2 contains some basic results which shall be used in later sections. In section 4.3, we prove $O(\varphi)$-inverse theorem for the operators $S_n(\cdot,k,t)$ and the $o(\varphi)$-inverse theorem for $S_n(\cdot,k,t)$ is proved in section 4.4. Similarly, $O(\varphi)$ and $o(\varphi)$-inverse theorems for the operators $S_{n,m}(\cdot,t)$ are proved, respectively, in sections 4.5 and 4.6.

4.2 Basic Results.

This section contains results which are useful in the later sections of this chapter. Throughout the rest of this chapter, we denote $I = [A,B]$, $I_j = [a_j,b_j]$, $j = 1,2,3,$ where $A < a_j < a_{j+1} < b_{j+1} < b_j < B$.

The following theorem of Sinha [61], establishes the convergence of the operators $S_n(\cdot,t)$.

Theorem 4.2.1. Let $f \in L_p(I)$ $(1 \leq p < \infty)$. Then

$$||S_n(f,t) - f(t)||_{L_p[A,B]} = o(1) \quad (n \to \infty).$$

A formula for the moments of exponential type operators [39, 1] is given by the following

<u>Lemma</u> 4.2.2. For $m \in \mathbb{N}$,

$$A_m(n,t) = n^m \int_A^B W(n,t,u)\,(u-t)^m\,du$$

is a polynomial in t of degree $\leq m$, in n of degree $[\frac{m}{2}]$. The coefficient of n^m in $A_{2m}(n,t)$ is $(2m-1)!!\ p^m(t)$ and the coefficient of n^m in $A_{2m+1}(n,t)$ is $C_m p^m(t)p'(t)$ where $C_m = \frac{(2m+1)!!}{3}\ m$.

<u>Corollary</u> 4.2.3. Let r be a positive real number. Then, for all t belonging to a compact subset K of (A,B) there holds

$$|A_r(n,t)| \leq M\ n^{r/2} \quad ;$$

M is a constant independent of n.

Sinha [61] obtained the following lemma regarding moment expansion of the operators $S_{n,m}(\cdot,t)$.

<u>Lemma</u> 4.2.4. Let K be a compact subset of (A,B). Then, for $k \in \mathbb{N}$ there hold :

1. If $k \leq m$, $S_{n,m}((u-t)^k,t) = 0$.

2. $$S_{n,m}((u-t)^{m+1},t) = (-1)^m \quad S_n(\prod_{i=0}^{m}(u-t+\frac{i}{n^{1/2}});\ t) + O(\frac{1}{n^{m+\frac{3}{2}}}) \quad (n \to \infty),$$

uniformly in $t \in K$.

3. If $k > m+1$,

$$S_{n,m}((u-t)^k;t) = \sum_{r=0}^{k-1} \frac{a_r}{n^{r/2}} S_n((u-t)^{k-r};t)$$

$$+ O(n^{-(m+k+2)/2} \quad (n \to \infty),$$

uniformly in $t \in K$ where a_r's are certain constants.

The following is an immediate corollary of the above lemma

Corollary 4.2.5.

$$S_{n,m}((u-t)^{m+1};t) = (-1)^m \frac{Q_{m+1}(t)}{n^{(m+1)/2}} + o(\frac{1}{n^{(m+1)/2}})$$

as $n \to \infty$, where $Q_{m+1}(t)$ is a polynomial in t of degree $\leq m+1$ and $Q_{m+1}(t) > 0$ in (A,B). The o-term holds uniformly with respect to $t \in K$.

Definition 4.2.6. A positive function $\psi \in C(A,B)$ is said to be a growth test function for $\{S_n\}$ if for any compact subset K of (A,B) there exists an $n_o \in \mathbb{N}$ and a positive constant M such that

$$S_n(\psi^2;t) < M, \quad n > n_o, \quad t \in K.$$

Lemma 4.2.7. Let $f \in L_p[A,B](1 \leq p < \infty)$. Then, for some constant M

$$\|S_{n,m}(f;t)\|_{L_p[A,B]} \leq M\|f\|_{L_p[A,B]} .$$

Regarding the convergence of the operator $S_{n,m}(\cdot,t)$, Sinha [61] proved the following

Theorem 4.2.8. Let $f \in L_p[A,B](1 \leq p < \infty)$. Then

$$\|S_{n,m}(f,t) - f(t)\|_{L_p[A,B]} = o(1) \quad (n \to \infty).$$

May [39] proved the following asymptotic formula for linear combinations of exponential type operators.

Lemma 4.2.9. Let $|f(t)| \leq \psi(t)$, $t \in (A,B)$ for some growth test function ψ. If for some $t \in (A,B)$, $f^{(2k+2)}(t)$ exists then

$$(4.2.1) \quad n^{k+1}\{S_n(f,k,t)-f(t)\} = \sum_{j=k+1}^{2k+2} Q(j,k,t)f^{(j)}(t) + o(1),$$

where $Q(j,k,t)$'s are certain polynomials in t. Moreover, $Q(2k+2,k,t) = c_1(p(t))^{k+1}$ and $Q(2k+1,k,t) = c_2(p(t))^k p'(t)$, c_1, c_2 being constants. Further, if $f \in C^{2k+2}[A,B]$, then (4.2.1) is uniform in every interval $[a,b] \subset (A,B)$.

Sinha [61] proved a similar result about the operators $S_{n,m}(\cdot,t)$.

Theorem 4.2.10. Let $f \in C_0^{m+1}$ with supp $f \subset I_1$. Then

$$S_{n,m}(f;t)-f(t) = \frac{(-1)^m Q_{m+1}(t)}{(m+1)!n^{(m+1)/2}} f^{(m+1)}(t) + o\left(\frac{1}{n^{(m+1)/2}}\right) \qquad (n \to \infty)$$

and

$$S_{n,m+1}(f;t) - f(t) = o\left(\frac{1}{n^{(m+1)/2}}\right) \qquad (n \to \infty),$$

uniformly in $t \in I_1$, where $Q_{m+1}(t)$ is a polynomial in t of degree $\leq m+1$ and $Q_{m+1}(t) > 0$ on (A,B).

We state here a technical lemma of Sinha [61] , which is of use in our inverse theorem.

Lemma 4.2.11. Let $h \in L_p[A,B]$ $(1 \leq p < \infty)$. Then for any fixed positive number ℓ,

1. $$||S_{n,m}(|u-t|\ |h(u)|\ ;\ t)||_{L_p(I_2)}$$
$$\leq M_1\{n^{-1/2}||h||_{L_p(I_1)} + n^{-\ell}||h||_{L_p[A,B]}\}$$

and if, in addition, supp $h \subset (A,B)$, then for $i \in \mathbb{N}^0$

2. $$||S_n(|u-t|^i\ |h(u)|;t)||_{L_p(I_2)}$$
$$\leq M_2\{n^{-i/2}||h||_{L_p(I_1)} + n^{-\ell}||h||_{L_p[A,B]}\},$$

where M_1, M_2 are constants independent of n and h.

Sinha [61] obtained Bernstein-type inequalities for the operators $S_n(\cdot,t)$ and $S_{n,m}(\cdot,t)$.

Lemma 4.2.12. Let $f \in L_p[A,B]$ $(1 \leq p < \infty)$ with supp $f \subset I_1$. Then

$$||S_n^{(2k+2)}(f;t)||_{L_p(I_1)} \leq M\,n^{k+1}||f||_{L_p(I_1)}\,.$$

In addition, if f has $2k+2$ derivatives with $f^{(2k+1)} \in A.C.(I_1)$ and $f^{(2k+2)} \in L_p(I_1)$ then

$$||S_n^{(2k+2)}(f;t)||_{L_p(I_1)} \leq M_1||f^{(2k+2)}||_{L_p(I_1)},$$

where the constants M,M_1 are independent of n and f.

Lemma 4.2.13. Let $f \in L_p[A,B]$ $(1 \le p < \infty)$ with supp $f \subset I_1$. Then

$$\|\bar{S}_{n,m}^{(m+1)}(f,t)\|_{L_p(I_1)} \le M\, n^{\frac{m+1}{2}} \|f\|_{L_p(I_1)},$$

Moreover, if f has $m+1$ derivatives over I_1 with $f^{(m)} \in A.C.(I_1)$ and $f^{(m+1)} \in L_p(I_1)$ then

$$\|\bar{S}_{n,m}^{(m+1)}(f;t)\|_{L_p(I_1)} \le M_1 \|f^{(m+1)}\|_{L_p(I_1)}$$

where

$$\bar{S}_{n,m}(f;t) = \int_A^B W(n,t,u) \{ \sum_{j=0}^{m} \frac{n^{j/2}}{j!} \times$$

$$\times \left(\prod_{i=0}^{j-1} \left(t-u-\frac{i}{n^{1/2}}\right)\right) \Delta^j f(u) \} \, du.$$

Lemma 4.2.14. Let $1 \le p < \infty$, $h \in L_p[A,B]$ and $i,j \in \mathbb{N}^0$. Then for any fixed positive number ℓ,

$$\|S_n(|u-t|^i \, |\int_t^u |u-w|^j \, |h(w)| \, dw| ;t)\|_{L_p(I_2)}$$

$$\le M\{n^{-(i+j+1)/2} \|h\|_{L_p(I_1)} + n^{-\ell} \|h\|_{L_p(I)} \}.$$

Furthermore, if $-\infty < A < B < \infty$, then

$$\|S_n(|u-t|^i \, |\int_t^u |u-w|^j \, |h(w)| dw| ;t)\|_{L_p[A,B]}$$

$$\le M_1 \, n^{-(i+j+1)/2} \|h\|_{L_p[A,B]} .$$

<u>Lemma</u> 4.2.15. Let $h \in B.V.(I_1)$ and $X(u)$ be the characteristic function of I_1. Then, for $i,j \in \mathbb{N}^0$, there holds

$$||S_n(X(u)|u-t|^i| \int_t^u |u-w|^j |dh(w)||;t)||_{L_1(I_2)}$$
$$\leq \frac{M_1}{n^{(i+j+1)/2}} ||h||_{B.V.(I_1)} .$$

Furthermore, if $A,B \in \mathbb{R}$ and $h \in B.V.(I)$, then

$$||S_n(|u-t|^i \int_t^u |u-w|^j |dh(w)||;t)||_{L_1(I)}$$
$$\leq M_2\, n^{-(i+j+1)/2} ||h||_{B.V.(I)}$$

In above, M and M_1 are constants.

We combine the direct theorems of Sinha [61] for the operators $S_n(\cdot,k,t)$ and $S_{n,m}(\cdot,t)$ in the following

<u>Theorem</u> 4.2.16. Let $f \in L_p[A,B]$ $(1 \leq p < \infty)$. Then for all sufficiently large values of n,

1. $$||S_n(f,k,t) - f(t)||_{L_p(I_2)}$$
$$\leq M_1\{\omega_{2k+2}(f,n^{-1/2},p,I_1) + n^{-(k+1)}||f||_{L_p[A,B]}\}$$

and

2. $$||S_{n,m}(f;t) - f(t)||_{L_p(I_2)}$$
$$\leq M_2\{\omega_{m+1}(f,n^{-1/2},p,I_1) + n^{-(m+2)/2}||f||_{L_p(I)}\},$$

where M_1, M_2 are constants.

We close this section by stating inverse theorems of Sinha [61] , regarding $S_n(\cdot,k,t)$ and $S_{n,m}(\cdot,t)$.

Theorem 4.2.17. Let $0 < \alpha < 2k+2$ and $f \in L_p[A,B]$ $(1 \leq p <$
Then

$$||S_n(f,k,t) - f(t)||_{L_p(I_1)} = O(n^{-\alpha/2}) \qquad (n \to \infty)$$

implies that

$$\omega_{2k+2}(f,t,p,I_2) = O(t^{\alpha}) \qquad (t \to 0).$$

Theorem 4.2.18. Let $0 < \alpha < m+1$ and $f \in L_r[A,B]$ $(1 \leq p < \infty)$.
Then

$$||S_{n,m}(f;t) - f(t)||_{L_p(I_1)} = O(n^{-\alpha/2}) \qquad (n \to \infty)$$

implies that

$$\omega_{m+1}(f,t,p,I_2) = O(t^{\alpha}) \qquad (t \to 0).$$

4.3 $O(\varphi)$-inverse theorem for $S_n(\cdot,k,t)$.

Theorem 4.2.16 implies that, if $\omega_{2k+2}(f,t,p,I_1) = O(\varphi(t))$ $(t \to 0)$, then $S_n(\cdot,k,t) = O(\varphi(n^{-1/2}))$ $(n \to \infty)$. Now, in this section, we prove the corresponding $O(\varphi)$-inverse theorem.

Theorem 4.3.1. Let $\varphi \in \Phi_{2k+2}$ and $f \in L_p[A,B]$ $(1 \leq p < \infty)$.
Then

$$(4.3.1) \qquad ||S_n(f,k,t) - f(t)||_{L_p(I_1)} = O(\varphi(n^{-1/2})) \qquad (n \to \infty)$$

implies that

$$(4.3.2)\qquad \omega_{2k+2}(f,t,p,I_2) = O(\varphi(t)) \quad (t \to 0).$$

Proof. Let (x_i, y_i), $i=1,2,3,4$ be such that $a_1 < x_1 < x_2 < x_3 < x_4 < a_2 < b_2 < y_4 < y_3 < y_2 < y_1 < b_1$.

Case 1. (For functions with a restricted support): Suppose supp $f \subsetneq (x_3, y_3)$. Then

$$\|\Delta_\gamma^{2k+2} f(t)\|_{L_p[x_3,y_3]} \le \|\Delta_\gamma^{2k+2}\{f(t)-S_n(f,k,t)\}\|_{L_p[x_3,y_3]} + \|\Delta_\gamma^{2k+2} S_n(f,k,t)\|_{L_p[x_3,y_3]}.$$

Proceeding as in the proof of theorem 2.3.1, for sufficiently small η, $\gamma > 0$, we get

$$\|\Delta_\gamma^{2k+2} S_n(f,k,t)\|_{L_p[x_3,y_3]} \le \gamma^{2k+2}\|S_n^{(2k+2)}(f,k,t)\|_{L_p[x_2,y_2]}$$

$$\le \gamma^{2k+2}\{S_n^{(2k+2)}(f-f_{\eta,2k+2},k,t)\|_{L_p[x_2,y_2]} + \|S_n^{(2k+2)}(f_{\eta,2k+2},k,t)\|_{L_p[x_2,y_2]}\}.$$

Hence, from lemmas 4.2.12 and 0.6.5 , we get

$$(4.3.3)\qquad \|\Delta_\gamma^{2k+2} S_n(f,k,t)\|_{L_p[x_3,y_3]} \le M_1\gamma^{2k+2}\left(n^{k+1}+\frac{1}{\eta^{2k+2}}\right)\omega_{2k+2}(f,\eta,p,[x_3,y_3]).$$

Now, since supp $f \subsetneq (x_3, y_3)$ and γ is sufficiently small, by hypothesis we get that

$$(4.3.4)\quad \|\Delta_\gamma^{2k+2}\{S_n(f,k,t)-f(t)\}\|_{L_p[x_3,y_3]} = O(\varphi(n^{-1/2})) \quad (n \to \infty$$

Thus, from (4.3.3-4), we have

$$||\Delta_{\gamma}^{2k+2} f(t)||_{L_p[x_3,y_3]} \leq M_2\{\varphi(n^{-1/2}) + \gamma^{2k+2}(n^{k+1} + \frac{1}{\eta^{2k+2}})$$

$$\times\, \omega_{2k+2}(f,\eta,p,[x_3,y_3])\}.$$

Choosing n such that $n \leq \eta^{-2} < 2n$, we observe that $\varphi(n^{-1/2}) \leq K_{2^{-1/2}}\, \varphi(\eta)$ and hence

$$||\Delta_{\gamma}^{2k+2} f(t)||_{L_p[x_3,y_3]}$$

$$\leq M_3\{\varphi(\eta) + (\frac{\gamma}{\eta})^{2k+2}\, \omega_{2k+2}(f,\eta,p,[x_3,y_3])\}.$$

This being true for all γ sufficiently small, taking supremum over all $\gamma \leq t$, we get

$$\omega_{2k+2}(f,t,p,[x_3,y_3])$$

$$\leq M_4\{\varphi(\eta) + (\frac{t}{\eta})^{2k+2}\, \omega_{2k+2}(f,\eta,p,[x_3,y_3]\},$$

which by lemma 1.2.2 implies that

$$\omega_{2k+2}(f,t,p,[x_3,y_3]) = O(\varphi(t)) \qquad (t \to 0).$$

The conclusion, now, follows since $I_2 \subset (x_3,y_3)$.

<u>Case 2</u>: (The general case) Now, we prove the theorem for general f.

Let $g \in C_0^{2k+2}$ such that supp $g \subset (x_3,y_3)$ and $g(t) = 1$ for $t \in [x_4,y_4]$.

In view of the case 1, it suffices to prove that

$$(4.3.5) \qquad \|fg(t) - S_n(fg,k,t)\|_{L_p[x_3,y_3]} = O(\varphi(n^{-1/2})) \quad (n \to \infty).$$

Let $\varphi \in \Phi_1$. Then, for some ξ lying between u and t,

$$\|S_n(fg,k,t) - (fg)(t)\|_{L_p[x_3,y_3]}$$

$$\leq \|S_n((f(u)-f(t))g(t),k,t)\|_{L_p[x_3,y_3]}$$

$$+ \|S_n(f(u)(g(u)-g(t)),k,t)\|_{L_p[x_3,y_3]}.$$

$$\leq M_4\,\varphi(n^{-1/2}) + \|S_n(f(u)(u-t)g'(\xi),k,t)\|_{L_p[x_3,y_3]}.$$

Thus, from hypothesis (4.3.1) and lemma 4.2.11, we get

$$\|S_n(fg,k,t)-(fg)(t)\|_{L_p[x_3,y_3]}$$

$$\leq M_5\{\varphi(n^{-1/2}) + n^{-1/2}\}$$

$$\leq M_6\,\varphi(n^{-1/2}),$$

since $\varphi \in \Phi_1$. Thus, the theorem is proved for all $\varphi \in \Phi_1$.

Also, we observe that $\Phi_1 \subset \Phi_2 \subset \cdots \subset \Phi_{2k+2}$. Hence, to complete the proof of the theorem, it is sufficient to prove (4.3.5) for all $\varphi \in \Phi_{r+1}$, assuming the theorem for all $\varphi \in \Phi_r$ where $r \in \mathbb{N}$ is such that $1 \leq r \leq 2k+1$. Let $\varphi \in \Phi_{r+1}$. We have

$$\|S_n(fg,k,t) - (fg)(t)\|_{L_p[x_3,y_3]}$$

$$\leq \|S_n((f(u)-f(t))g(t),k,t)\|_{L_p[x_3,y_3]} +$$

$$+ \|S_n(f(u)(g(u)-g(t)),k,t)\|_{L_p[x_3,y_3]}$$

$$\leq M_7\,\varphi(n^{-1/2}) + \|S_n(f(u)(g(u)-g(t)),k,t)\|_{L_p[x_3,y_3]}$$

$$\leq M_7\,\varphi(n^{-1/2}) +$$

$$+ \|S_n((f(u)-f_{\eta,2k+2}(u))(g(u)-g(t)),k,t)\|_{L_p[x_3,y_3]}$$

$$+ \|S_n((f_{\eta,2k+2}(u)-f_{\eta,2k+2}(t))(g(u)-g(t)),k,t)\|_{L_p[x_3,y_3]}$$

$$+ \|S_n(f_{\eta,2k+2}(t)(g(u)-g(t)),k,t)\|_{L_p[x_3,y_3]}$$

(4.3.6) $= M_7\,\varphi(n^{-1/2}) + Z_1+Z_2+Z_3$, say.

By lemmas 4.2.9 and 0.6.5

(4.3.7) $\quad Z_3 \leq \dfrac{M_8}{n^{k+1}}\|f\|_{L_p(I)}$.

For some ξ lying between u and t,

$$Z_1 = \|S_n((f(u)-f_{\eta,2k+2}(u))(u-t)g'(\xi),k,t)\|_{L_p[x_3,y_3]}.$$

Now, using the definition of the linear combinations and lemmas 4.2.11 and 0.6.5 (taking $\ell = k+1$) , we get

(4.3.8) $\quad Z_1 \leq M_9\{n^{-1/2}\,\omega_{2k+2}(f,\eta,p,[x_1,y_1]) + n^{-(k+1)}\|f\|_{L_p(I)}\}$

For some ξ lying between u and t,

$$Z_2 \leq \sum_{i=1}^{2k+1}\sum_{j=1}^{2k}\frac{1}{i!j!}\|f^{(i)}_{\eta,2k+2}(t)g^{(j)}(t)S_n((u-t)^{i+j},k,t)\|_{L_p[x_3,y_3]}+$$

$$+ \frac{1}{(2k+1)!} \{ \sum_{i=1}^{2k+1} \frac{1}{i!} \| S_n(f_{\eta,2k+2}^{(i)}(t) g^{(2k+1)}(\xi) \times$$

$$\times (u-t)^{2k+1+i}, k, t) \|_{L_p[x_3,y_3]}$$

$$+ \frac{1}{(2k+1)!} \| S_n((\sum_{i=1}^{2k} \frac{g^{(i)}(t)}{i!} (u-t)^i) \times$$

$$\times (\int_t^u (u-w)^{2k+1} f_{\eta,2k+2}^{(2k+2)}(w) dw), k, t) \|_{L_p[x_3,y_3]}$$

$$+ \frac{1}{((2k+1)!)^2} \| S_n((u-t)^{2k+1} g^{(2k+1)}(\xi) \times$$

$$\times \int_t^u (u-w)^{2k+1} f_{\eta,2k+2}^{(2k+2)}(w) dw, k, t) \|_{L_p[x_3,y_3]}$$

(4.3.9) $\quad = R_1 + R_2 + R_3 + R_4$, say.

Using lemmas 4.2.14-15, (taking $\ell = 2k+2$), we have

(4.3.10) $\quad R_3 \leq M_{10} \{ n^{-(k+3/2)} \| f_{\eta,2k+2}^{(2k+2)} \|_{L_p[x_2,y_2]}$

$$+ n^{-(2k+2)} \| f_{\eta,2k+2}^{(2k+2)} \|_{L_p(I)} \}$$

and

(4.3.11) $\quad R_4 \leq M_{11} \{ n^{-(2k+3/2)} \| f_{\eta,2k+2}^{(2k+2)} \|_{L_p[x_2,y_2]}$

$$+ n^{-(2k+2)} \| f_{\eta,2k+2}^{(2k+2)} \|_{L_p(I)} \}.$$

Since $\sum_{j=0}^{k} C(j,k) d_j^{-m} = 0$, $m = 1,2,\ldots,k$, from lemma 4.2.2, it follows that

$$R_1 \leq \frac{M_{12}}{n^{k+1}} \left(\sum_{i=1}^{2k+1} \|f^{(i)}_{\eta,2k+2}\|_{L_p[x_3,y_3]} \right)$$

and by corollary 4.2.3, we get

$$R_2 \leq \frac{M_{13}}{n^{k+1}} \left(\sum_{i=1}^{2k+1} \|f^{(i)}_{\eta,2k+2}\|_{L_p[x_3,y_3]} \right).$$

Now applying lemma C.6.3 , we get

$$(4.3.12) \quad R_1, R_2 \leq \frac{M_{14}}{n^{k+1}} \{ \|f^{(2k+1)}_{\eta,2k+2}\|_{L_p[x_3,y_3]} + \|f_{\eta,2k+2}\|_{L_p[x_3,y_3]} \}.$$

Choosing n such that $n \leq \eta^{-2} < 2n$, from (4.3.7-12) and lemma C.6.5

$$(4.3.13) \quad Z_1+Z_2+Z_3 \leq M_{15}\{ n^{-1/2}\, \omega_{2k+2}(f,n^{-1/2},p,[x_1,y_1])$$
$$+ n^{-1/2}\, \omega_{2k+1}(f,n^{-1/2},p,[x_1,y_1]) + n^{-(k+1)} \|f\|_{L_p(I)} \}$$

We can observe that, since $\varphi \in \Phi_{r+1}$,

$$\varphi^*(t) = \frac{\varphi(t)}{t} \in \Phi_r .$$

Now, since the theorem is true for all $\varphi \in \Phi_r$, we have

$$(4.3.14) \quad \omega_{2k+2}(f,t,p,[x_1,y_1]) = O(\varphi^*(t)) \quad (t \to 0).$$

Also, lemma 1.?.4 and the fact that $\varphi^* \in \Phi_{2k+1}$ imply that

$$(4.3.15) \quad \omega_{2k+1}(f,t,p,[x_1,y_1]) = O(\varphi^*(t)) \quad (t \to 0).$$

Thus, (4.3.5) follows from (4.3.13-15) and the theorem is proved.

<u>Corollary</u> 4.3.2. Let $1 \leq m \leq 2k+2$, $\varphi \in \Phi_m$ and $f \in L_p(I)(1 \leq p < \infty)$. Then

$$||S_n(f,k,t) - f(t)||_{L_p(I_1)} = O(\varphi(n^{-1/2})) \quad (n \to \infty)$$

implies that

$$\omega_m(f,t,p,I_2) = O(\varphi(t)) \quad (t \to 0).$$

<u>Corollary</u> 4.3.3. Theorem 4.2.17.

4.4 <u>$o(\varphi)$-inverse theorem for $S_n(\cdot,k,t)$</u>

It follows from theorem 4.2.16 that, if $\omega_{2k+2}(f,t,p,I_1) = o(\varphi(t))$ $(t \to 0)$ then $||S_n(f,k,t) - f(t)||_{L_p(I_2)} = o(\varphi(n^{-1/2}))$ $(n \to \infty)$.
The $o(\varphi)$-inverse theorem for the operator $S_n(\cdot,k,t)$ is as follows:

<u>Theorem</u> 4.4.1. Let $f \in L_p(I)$ $(1 \leq p < \infty)$ and $\varphi \in \Phi_{2k+2}$. Then

$$(4.4.1) \qquad ||S_n(f,k,t) - f(t)||_{L_p(I_1)} = o(\varphi(n^{-1/2})) \quad (n \to \infty)$$

implies that

$$(4.4.2) \qquad \omega_{2k+2}(f,t,p,I_2) = o(\varphi(t)).$$

<u>Proof</u>. Let x_i, y_i be such that, for $i=1,2,3$,
$a_1 < x_i < x_i+1 < a_2 < b_2 < y_{i+1} < y_i < b_1$.

Case 1. (For functions with restricted support).

Suppose supp $f \subset (x_3, y_3)$. Then proceeding as in the proof of theorem 4.3.1, for sufficiently small $\gamma, \eta > 0$, we get

$$(4.4.3) \quad \|\Delta_\gamma^{2k+2} f(t)\|_{L_p[x_3,y_3]} \le M_1 \{\varphi(n^{-1/2}) + \\ + \gamma^{2k+2}(n^{k+1} + \frac{1}{\eta^{2k+2}})\, \omega_{2k+2}(f, \eta, p, [x_3, y_3])\}.$$

Now, we define $\psi(x)$ as follows:

$$\psi(x) = \begin{cases} \|\Delta_\gamma^{2k+2}\{f(t) - S_n(f,k,t)\}\|_{L_p[x_3,y_3]} & \text{if } x = n^{-1/2} \\ \psi(n^{-1/2}) & \text{if } x \in ((n+1)^{-1/2}, n^{-1/2}) \end{cases}$$

for $n = 1, 2, \ldots$.

Since γ is sufficiently small and supp $f \subset (x_3, y_3)$, by hypothesis (4.4.1), we get that

$$(4.4.4) \quad \|\Delta_\gamma^{2k+2}\{f(t) - S_n(f,k,t)\}\|_{L_p[x_3,y_3]} = o(\varphi(n^{-1/2})) \quad (n \to \infty).$$

Also, if $t \in ((n+1)^{-1/2}, n^{-1/2})$ then

$$\frac{\psi(t)}{\varphi(t)} \le K_{2^{-1/2}} \frac{\psi(n^{-1/2})}{\varphi(n^{-1/2})}$$

and hence $\psi(t) = o(\varphi(t)) \quad (t \to 0)$.

Thus, from (4.4.3-4), we get

$$\|\Delta_\gamma^{2k+2} f(t)\|_{L_p[x_3,y_3]} \le M_2 \{\psi(n^{-1/2}) + \gamma^{2k+2}(n^{k+1} + \frac{1}{\eta^{2k+2}}) \times \\ \times \omega_{2k+2}(f, \eta, p, [x_3, y_3])\}.$$

Taking supremum over $\gamma \leq t$ and choosing n such that $n \leq \eta^{-2} < n+1$, we get

$$\omega_{2k+2}(f,t,p,[x_3,y_3])$$
$$\leq M_3\{\psi(\eta) + (t/\eta)^{2k+2}\,\omega_{2k+2}(f,\eta,p,[x_3,y_3])\},$$

which in view of lemma 1.2.3 proves the theorem for case 1.

Case 2. (general functions).

Let $g \in C_o^{2k+2}$ such that $\operatorname{supp} g \subset (x_3,y_3)$ and $g(t) = 1$ for $t \in [x_4,y_4]$.

Again, proceeding as in the proof of theorem 4.3.1, we are left to show that

$$(4.4.5) \qquad ||fg(t) - S_n(fg,k,t)||_{L_p[x_3,y_3]} = o(\varphi(n^{-1/2})) \quad (n \to \infty).$$

Let $\varphi \in \Phi_1$. Then, clearly as for theorem 4.3.1, it follows that

$$||(fg)(t)-S_n(fg,k,t)||_{L_p[x_3,y_3]} = o(\varphi(n^{-1/2})) \quad (n \to \infty)$$

and hence the theorem is proved for all $\varphi \in \Phi_1$.
Next, assume that the result holds for all $\varphi \in \Phi_r$ $(1 \leq r \leq 2k+1)$ and let $\varphi \in \Phi_{r+1}$. Again proceeding as in theorem 4.3.1, we get

$$||S_n((fg),k,t) - (fg)(t)||_{L_p[x_3,y_3]} = Z_o + Z_1 + Z_2 + Z_3,$$

where

$$Z_o = ||S_n((f(u)-f(t))g(t),k,t)||_{L_p[x_3,y_3]} = o(\varphi(n^{-1/2})) \quad (n\to\infty),$$

Z_1, Z_2, Z_3 are as before and satisfy

$$(4.4.6) \quad Z_1+Z_2+Z_3 \leq M_4\{n^{-1/2}\,\omega_{2k+2}(f,n^{-1/2},p,[x_1,y_1])$$

$$+ n^{-1/2}\,\omega_{2k+1}(f,n^{-1/2},p,[x_1,y_1]) + n^{-(k+1)}||f||_{L_p(I)}\}$$

Now, defining $\varphi^*(t) = \varphi(t)/t$, we observe that $\varphi^* \varepsilon \Phi_r$. Hence by the assumption, we get

$$(4.4.7) \quad \omega_{2k+2}(f,t,p,[x_1,y_1]) = o(\varphi^*(t)) \qquad (t\to 0),$$

which from lemma 1.2.5 and the fact that $\varphi^* \varepsilon \Phi_{2k+1}$ implies that

$$(4.4.8) \quad \omega_{2k+1}(f,t,p,[x_1,y_1]) = o(\varphi^*(t)) \quad (t\to 0).$$

Thus, from (4.4.6-4.4.8), we get (4.4.5) and hence the theorem for all $\varphi \varepsilon \Phi_{r+1}$.

Thus, we have proved the theorem, first, for all $\varphi \varepsilon \Phi_1$. Later, assuming the theorem for $\varphi \varepsilon \Phi_r$ $(1\leq r\leq 2k+1)$, we proved it for $\varphi \varepsilon \Phi_{r+1}$.

The theorem, now, follows for all $\varphi \varepsilon \Phi_{2k+2}$, since $\Phi_1 \subset \Phi_2 \subset \cdots \subset \Phi_{2k+2}$.

As in section 4.3, we get an immediate

<u>Corollary</u> 4.4.2. Let $f \varepsilon L_p(I)$ and $\varphi \varepsilon \Phi_r$ for some $1 \leq r \leq 2k+2$. Then

$$\|S_n(f,k,t) - f(t)\|_{L_p(I_1)} = O(\varphi(n^{-1/2})) \quad (n \to \infty)$$

implies that

$$\omega_r(f,t,p,I_2) = o(\varphi(t)) \quad (t \to 0).$$

4.5 <u>$O(\varphi)$-inverse theorem for $S_{n,m}(\cdot,t)$</u>

Theorem 4.2.16 implies that if

$\omega_{m+1}(f,t,p,I_1) = O(\varphi(t)) \quad (t \to 0)$ then

$$\|S_{n,m}(f;t) - f(t)\|_{L_p(\bar{I}_2)} = O(\varphi(n^{-1/2})) \quad (n \to \infty).$$

Now, the corresponding inverse theorem is as follows:

<u>Theorem</u> 4.5.1. Let $f \in L_p(\bar{I})$ and $\varphi \in \Phi_{m+1}$. Then

$$\|S_{n,m}(f;t) - f(t)\| = O(\varphi(n^{-1/2})) \quad (n \to \infty)$$

implies that

$$\omega_{m+1}(f,t,p,\bar{I}_2) = O(\varphi(t)) \quad (t \to 0).$$

<u>Proof</u>. Let x_i, y_i, $i=1,2,3,4$ satisfy

$$a_1 < x_1 < x_2 < x_3 < x_4 < a_2 < b_2 < y_4 < y_3 < y_2 < y_1 < b_1.$$

Suppose supp $f \subset (x_3,y_3)$. Defining

$$F(t,u) = \sum_{j=0}^{m} \left\{\frac{n^{j/2}}{j!}\left(\prod_{i=0}^{j-1}\left(t-u-\frac{i}{n^{1/2}}\right)\right) \Delta^j f(u)\right\}$$

and

$$\bar{S}_{n,m}(f;t) = \int_A^B w(n,t,u)\, F(t,u)\,du,$$

we observe that

$$S_{n,m}(f;t) = \bar{S}_{n,m}(f;t) - \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} |u-t|^{m_o+2} \times$$

$$\times F(t,u)du.$$

Hence, for sufficiently small values of γ , we have

$$||\Delta_\gamma^{m+1} f(t)||_{L_p[x_3,y_3]} \leq ||\Delta_\gamma^{m+1}\{f(t)-S_{n,m}(f;t)\}||_{L_p[x_3,y_3]}$$

$$+ ||\Delta_\gamma^{m+1} S_{n,m}(f;t)||_{L_p[x_3,y_3]}$$

$$\leq ||\Delta_\gamma^{m+1}\{f(t)-S_{n,m}(f;t)\}||_{L_p[x_3,y_3]}$$

$$+ ||\Delta_\gamma^{m+1} \bar{S}_{n,m}(f;t)||_{L_p[x_3,y_3]}$$

$$+ ||\Delta_\gamma^{m+1} \{ \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} |u-t|^{m_o+2} F(t,u)du\}||_{L_p[x_3,y_3]}$$

(4.5.1) $$= J_1+J_2+J_3, \quad \text{say} .$$

Proceeding as in theorem 2.3.1 and using lemmas 4.2.13 and 0.6.5, we get

(4.5.2) $$J_2 \leq M_2 \gamma^{m+1} (n^{\frac{m+1}{2}} + \eta^{-(m+1)}) \omega_{m+1}(f,\eta,p,[x_2,y_2]).$$

To obtain an estimate for J_3, consider

$$T_1(t) = cn^{\frac{(j-r)}{2}} \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} |u-t|^{m_o+2} (t-u)^{j-r}(\Delta^j f(u))du,$$

where $0 \leq r \leq j-1$ and c is a constant.

Since supp $f \subset (x_3, y_3)$, lemma 4.2.11 implies that

$$||T_1(t)||_{L_p[x_3, y_3+(m+1)\gamma]} \leq \frac{M_3}{n^{(m+2)/2}} ||f||_{L_p[x_3, y_3]}$$

Consequently,

$$J_3 \leq \frac{M_4}{n^{(m+2)/2}} ||f||_{L_p[x_3, y_3]} . \tag{4.5.3}$$

Thus, from (4.5.1-3), we get

$$||\Delta_\gamma^{m+1} S_{n,m}(f;t)||_{L_p[x_3, y_3]}$$

$$\leq M_5 \, \gamma^{m+1} \{ (n^{(\frac{m+1}{2})} + \eta^{-(m+1)}) \, \omega_{m+1}(f, \eta, p, [x_2, y_2])$$

$$+ n^{-(m+2)/2} ||f||_{L_p[x_3, y_3]} \} .$$

Again, proceeding as in theorem 4.3.1, we get the result for all functions f with supp $f \subset (x_3, y_3)$. Also in the case of general support, we are left to show that

$$||(fg)(t) - S_{n,m}(fg;t)||_{L_p[x_3, y_3]} = O(\varphi(n^{-1/2})) \quad (n \to \infty), \tag{4.5.4}$$

where $g \in C_0^{m+1}$ such that supp $g \subset (x_3, y_3)$ and $g(t) = 1$ on $[x_4, y_4]$.

Now, we prove (4.5.4) by induction as follows:

First, we prove the theorem for all $\varphi \in \Phi_1$ and next assuming the theorem for all $\varphi \in \Phi_r$ $(1 \leq r \leq m)$, we prove the theorem for all $\varphi \in \Phi_{r+1}$.

Let $\varphi \varepsilon \Phi_1$. Then, for some ξ lying between u and t,

$$||S_{n,m}((fg);t) - (fg)(t)||_{L_p[x_3,y_3]}$$
$$\leq M_6\ \varphi(n^{-1/2}) + ||S_{n,m}(f(u)(u-t)g'(\xi);t)||_{L_p[x_3,y_3]}.$$

Now, using lemma 4.2.11 and the fact that $t = o(\varphi(t))$, we get (4.5.3) and hence the theorem is proved for all $\varphi \varepsilon \Phi_1$.

Now, assume that the theorem holds for all $\varphi \varepsilon \Phi_r$ $(1 \leq r \leq 2k+1)$. Let $\varphi \varepsilon \Phi_{r+1}$. Now

$$||S_{n,m}(fg;t) - (fg)(t)||_{L_p[x_3,y_3]}$$
$$\leq M_7\ \varphi(n^{-1/2}) + ||S_{n,m}((f(u)-f_{\eta,m+1}(u)) \times$$
$$\times\ (g(u)-g(t));t)||_{L_p[x_3,y_3]}$$
$$+ ||S_{n,m}((f_{\eta,m+1}(u) - f_{\eta,m+1}(t))\ (g(u)-g(t));t)||_{L_p[x_3,y_3]}$$
$$+ ||S_{n,m}((f_{\eta,m+1}(t))(g(u)-g(t));t)||_{L_p[x_3,y_3]}$$
$$= M_7\ \varphi(n^{-1/2}) + Z_1 + Z_2 + Z_3, \text{ say.}$$

For some ξ lying between u and t,

$$Z_1 \leq ||S_{n,m}((f(u)-f_{\eta,m+1}(u))\ (u-t)g'(\xi);t)||_{L_p[x_3,y_3]}$$

Now, using lemmas 4.2.11 and 0.6.5, we get

$$(4.5.5)\quad Z_1 \leq M_8\{n^{-1/2}\omega_{m+1}(f,\eta,p,[x_1,y_1]) + n^{-(m+1)/2}||f||_{L_p(I)}\}$$

Also, by theorem 4.2.10 and lemma 0.6.5

$$(4.5.6) \qquad Z_3 \leq \frac{M_9}{n^{(m+1)/2}} \, ||f||_{L_p(I)} .$$

Using Taylor expansions of $f_{\eta,m+1}$ and g we can write $Z_2 \leq \sum_{i=1}^{4} R_i$, where R_1, R_2, R_3, R_4 are as in theorem 2.3.1 with $P_{n,r}$ replaced by $S_{n,m}$.

Consider

$$T_2(t) = c\, f^{(i)}_{\eta,m+1}(t)\, n^{(j-r-s)/2}$$

$$\int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} (u-t)^{i+j+m-r-s} g^{(m)}(\xi_j)\, du,$$

where $1 \leq i \leq m$, $0 \leq j \leq m$, $0 \leq r \leq j-1$, $0 \leq s \leq m+i$, ξ_j lies between $u + \frac{i}{n^{1/2}}$ and t and c is a constant.

Applying corollary 4.2.3,

$$|T_2(t)| \leq \frac{M_{10}}{n^{(m+i)/2}} \, |f^{(i)}_{\eta,m+1}(t)| .$$

As, $T_2(t)$ is a typical component in R_2,

$$(4.5.7) \qquad R_2 \leq M_{11} \{ \sum_{i=1}^{m} n^{-(m+i)/2} ||f^{(i)}_{\eta,m+1}||_{L_p[x_3,y_3]} \}.$$

Now, from lemmas 4.2.4 and 4.2.2

$$(4.5.8) \qquad R_1 \leq M_{12} \sum_{\substack{i=1 \\ i+j>m}}^{m} \sum_{j=1}^{m-1} n^{-(i+j)/2} ||f^{(i)}_{\eta,m+1}||_{L_p[x_3,y_3]} .$$

A typical component of

$$S_{n,m}(g^{(m)}(\xi)(u-t)^k\{\int_t^u (u-w)^m f_{\eta,m+1}^{(m+1)}(w)\,dw\};t)$$

is of the form

$$T_3(t) = c\, n^{(\theta-k)/2} \int_A^B \psi(t,u)\,du$$

where

$$\psi(t,u) = \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} (u-t)^{\theta}\, g^{(m)}(\xi_{r_2}) \times$$

$$\times \int_t^{u+\frac{r_2}{n^{1/2}}} (u-w+\frac{r_2}{n^{1/2}})^m f_{\eta,m+1}^{(m+1)}(w)\,dw$$

and $\theta = j + r_j - r_1$, $0 \leq j \leq m$, $0 \leq r_1 \leq j-1$, $0 \leq r_2 \leq j$, $0 \leq r_3 \leq k$, ξ_{r_2} lies between $u+\frac{r_2}{n^{1/2}}$ and t and c is a constant.

Let $X(u)$ be the characteristic function of $[x_2, y_2]$. Then

$$T_3(t) = T_{31}(t) + T_{32}(t), \text{ say, where}$$

$$T_{31}(t) = c\, n^{(\theta-k)/2} \int_A^B X(u)\,\psi(t,u)\,du$$

and

$$T_{32}(t) = c\, n^{(\theta-k)/2} \int_A^B (1-X(u))\,\psi(t,u)\,du.$$

It follows from lemmas 4.2.14-15 that

$$||T_{31}(t)||_{L_p[x_3,y_3]} \leq \frac{M_{13}}{n^{(m+k+1)/2}} ||f^{(m+1)}_{\eta,m+1}||_{L_p[x_2,y_2]} .$$

It is easy to see that, for any $\ell > 0$

$$||T_{32}(t)||_{L_p[x_3,y_3]} \leq \frac{M_{14}}{n^{\ell}} ||f^{(m+1)}_{\eta,m+1}||_{L_p(I)} .$$

Thus,

$$||S_{n,m}(g^m(\xi)(u-t)^k(\int_t^u (u-w)^m f^{(m+1)}_{\eta,m+1}(w)dw);t)||_{L_p[x_3,y_3]}$$

$$\leq M_{15}\{n^{-(m+k+1)/2} ||f^{(m+1)}_{\eta,m+1}||_{L_p[x_2,y_2]}$$

$$+ n^{-\ell} ||f^{(m+1)}_{\eta,m+1}||_{L_p(I)}\}$$

and hence

$$(4.5.9) \quad R_3 \leq M_{16}\{n^{-(m+2)/2} ||f^{(m+1)}_{\eta,m+1}||_{L_p[x_2,y_2]}$$

$$+ n^{-\ell} ||f^{(m+1)}_{\eta,m+1}||_{L_p(I)}\}$$

and

$$(4.5.10) \quad R_4 \leq M_{17}\{n^{-(2m+1)/2} ||f^{(m+1)}_{\eta,m+1}||_{L_p[x_2,y_2]}$$

$$+ n^{-\ell} ||f^{(m+1)}_{\eta,m+1}||_{L_p(I)}\}$$

Taking $\ell = m+1$ and applying lemma 0.6.5 and combining (4.5.7-10), we get

$$(4.5.11)\quad Z_2 \le M_{18}\{n^{-(m+2)/2}\|f^{(m+1)}_{\eta,m+1}\|_{L_p[x_2,y_2]}$$

$$+ n^{-(m+1)/2}\|f^{(m)}_{\eta,m+1}\|_{L_p[x_3,y_3]}$$

$$+ n^{-(m+1)/2}\|f_{\eta,m+1}\|_{L_p[x_3,y_3]}$$

$$+ n^{-(m+1)}\|f^{(m+1)}_{\eta,m+1}\|_{L_p(I)}\}.$$

Applying lemma 0.6.5 and choosing n such that $n \le \eta^{-2} < 2n$, from (4.5.5-6,11), we get

$$Z_1+Z_2+Z_3 \le M_{19}\{n^{-1/2}\,\omega_{m+1}(f,n^{-1/2},p,[x_1,y_1])$$

$$+ n^{-1/2}\,\omega_m(f,n^{-1/2},p,[x_1,y_1])+ n^{-(m+1)/2}\|f\|_{L_p(I)}\}.$$

After this step, the rest of the proof is as in the case of theorem 4.3.1.

Corollary 4.5.2. Let $f \in L_p(I)$ and $\varphi \in \Phi_k$ for some k such that $1 \le k \le m+1$. Then

$$\|S_{n,m}(f;t) - f(t)\|_{L_p(\bar{I}_1)} = O(\varphi(n^{-1/2})) \qquad (n \to \infty)$$

implies that

$$\omega_k(f,t,p,\bar{I}_2) = O(\varphi(t)) \quad (t \to 0).$$

Taking $\varphi(t) = t^{\alpha}$ $(0 < \alpha < m+1)$, we get

Corollary 4.5.3. Theorem 4.2.18.

4.6 o(φ)-inverse theorem for $S_{n,m}(\cdot,t)$

In this section, we prove o(φ)-inverse theorem for the operator $S_{n,m}(\cdot,t)$ whose o(φ)-direct theorem is a consequence of Theorem 4.2.18.

Theorem 4.6.1. Let $f \in L_p(I)$ and $\varphi \in \Phi_{m+1}$. Then

$$(4.6.1) \qquad \|S_{n,m}(f,t)-f(t)\|_{L_p(I_1)} = o(\varphi(n^{-1/2})) \qquad (n \to \infty)$$

implies that

$$(4.6.2) \qquad \omega_{m+1}(f,t,p,I_2) = o(\varphi(t)) \qquad (t \to 0).$$

Proof. Let x_i, y_i be as in the proof of theorem 4.5.1. Suppose supp $f \subset (x_3,y_3)$ from (4.5.1-3) we get

$$\|\Delta_\gamma^{m+1} S_{n,m}(f,t)\|_{L_p[x_3,y_3]}$$

$$\leq M_1 \, \gamma^{m+1} \{(n^{(\frac{m+1}{2})} + \eta^{-(m+1)})\omega_{m+1}(f,\eta,p,[x_3,y_3])$$

$$+ n^{-(\frac{m+2}{2})} \|f\|_{L_p[x_3,y_3]} \}$$

Using the fact that $t^{m+1}/\varphi(t) \to 0$ as $t \to 0$,

we have

$$(4.6.3) \qquad \|\Delta_\gamma^{m+1} S_{n,m}(f,t)\|_{L_p[x_3,y_3]} \leq M_2 \{\gamma^{m+1}(n^{(\frac{m+1}{2})} + \eta^{-(m+1)}) \times$$

$$\times \omega_{m+1}(f,\eta,p,[x_3,y_3]) + \varphi(n^{-1/2})\}$$

By hypothesis (4.6.1) and since supp f $\subset (x_3, y_3)$, choosing γ sufficiently small, we get

(4.6.4) $$\|\Delta_\gamma^{2k+2}\{f(t)-S_{n,m}(f,k,t)\}\|_{L_p[x_3,y_3]} = o(\varphi(n^{-1/2})) \quad (n \to \infty).$$

Now defining

$$\psi(x) = \begin{cases} \|\Delta_\gamma^{2n+1}\{f(t)-S_{n,m}(f;t)\|_{L_p[x_3,y_3]} & \text{if } x = n^{-1/2} \\ \psi(n^{-1/2}) & \text{if } x \varepsilon ((n+1)^{-1/2}, n^{-1/2}), \end{cases}$$

for $n=1,2,\ldots,$ we observe that

$$\psi(x) = o(\varphi(x)) \quad (x \to 0).$$

Now, combining (4.6.3-4) and choosing n such that $n \leq \eta^{-2} < n+1$, we get

$$\|\Delta_\gamma^{m+1} f(t)\|_{L_p[x_3,y_3]} \leq M_3\{\psi(\eta) + (\tfrac{\gamma}{\eta})^{m+1} \omega_{m+1}(f,\eta,p,[x_3,y_3])\}.$$

Now, taking supremum over $\gamma \leq t$, we get

$$\omega_{m+1}(f,t,p,[x_3,y_3]) \leq M_4\{\psi(\eta) + (\tfrac{t}{\eta})^{m+1} \times$$

$$\times\, \omega_{m+1}(f,\eta,p,[x_3,y_3]\},$$

which proves the result by lemma 1.2.3. Thus, the theorem is proved for all f such that

$$\text{supp } f \subset (x_3, y_3).$$

Now we prove the theorem for general functions. We prove this by induction as follows: First, we prove the theorem for all $\varphi \varepsilon \Phi_1$. Next assuming the result of the theorem for all $\varphi \varepsilon \Phi_r$ for some r such that $1 \leq r \leq m$, we prove it for all $\varphi \varepsilon \Phi_{r+1}$.

Let $g \in C_o^{m+1}$ be such that supp $g \subset (x_3, y_3)$ and $g(t) = 1$ on $[x_4, y_4]$. If $\varphi \in \Phi_1$ then

$$||S_{n,m}(fg;t)-(fg)(t)||_{L_p[x_3,y_3]}$$

$$= ||S_{n,m}((f(u)-f(t))g(t);t)||_{L_p[x_3,y_3]}$$

$$+ ||S_{n,m}(f(u)(g(u)-g(t));t)||_{L_p[x_3,y_3]}$$

$$= ||g(t)\{S_{n,m}(f;t)-f(t)\}||_{L_p[x_3,y_3]}$$

$$+ ||S_{s,n}(f(u)(u-t)g'(\xi),t)||_{L_p[x_3,y_3]}$$

for some ξ lying between u and t.

Thus, from lemma 4.2.11 and using the fact that $t = o(\varphi(t))$, we get

$$||S_{n,m}(fg;t)-(fg)(t)||_{L_p[x_3,y_3]} = o(\varphi(n^{-1/2})) \quad (n \to \infty).$$

The result follows, now for all $\varphi \in \Phi_1$, since supp $fg \subset (x_3, y_3)$ and $fg = f$ on I_2.

Next, assume that the theorem holds for all $\varphi \in \Phi_r$ for some r such that $1 \leq r \leq 2k + 1$ and let $\varphi \in \Phi_{r+1}$. Then, from the proof of theorem 4.5.1, we get

$$||S_{n,m}(fg;t)-(fg)(t)||_{L_p[x_3,y_3]}$$

$$= ||g(t)\{S_{n,m}(f;t)-f(t)\}||_{L_p[x_3,y_3]}$$

$$+ ||S_{n,m}(f(u)(g(u)-g(t));t)||_{L_p[x_3,y_3]}$$

$$= o(\varphi(n^{-1/2})) + Z_1+Z_2+Z_3 ,$$

where $Z_1+Z_2+Z_3$ is estimated as

$$(4.6.5) \quad Z_1+Z_2+Z_3 \leq M\{n^{-1/2}\,\omega_{m+1}(f,n^{-1/2},p\,[x_1,y_1])$$

$$+ n^{-1/2}\,\omega_m\,(f,n^{-1/2},p[x_1,y_1]) + n^{-(\frac{m+1}{2})}||f||_{L_p(I)}\}$$

It is easy to see that $\varphi^* = \frac{\varphi(t)}{t} \varepsilon\, \Phi_r$. Hence, by induction hypothesis, we get

$$(4.6.6) \quad \omega_{m+1}(f,n^{-1/2},p,[x_1,y_1]) = o(\varphi^*(n^{-1/2}))\,(n \to \infty)$$

which coupled with lemma 1.2.5 implies that

$$(4.6.7) \quad \omega_m(f,n^{-1/2},p,[x_1,y_1]) = o(\varphi^*\,(n^{-1/2})) \quad (n \to \infty).$$

Combining (4.6.5-7) with the fact that $n^{-(\frac{m+1}{2})}/\varphi(n^{-1/2}) \to 0$ as $n \to \infty$, the result follows .

<u>Corollary</u> 4.6.2. Let $f \;\varepsilon\; L_p\,[A,B]$ $(1 \leq p < \infty)$ and $\varphi \;\varepsilon\; \Phi_r$ $(1 \leq r \leq m+1)$. Then

$$||S_{n,m}(f;t)-f(t)||_{L_p(I_1)} = o(\varphi\,(n^{-1/2})) \qquad (n \to \infty)$$

implies that

$$\omega_r(f,t,p,I_2) = o(\varphi(t)) \qquad (t \to 0)$$

CHAPTER V

φ-INVERSE THEOREMS FOR LINEAR COMBINATIONS OF GENERALISED JACKSON OPERATORS

5.1 Introduction.

In this chapter we study the L_q-approximation $(1 \le q < \infty)$ by linear combinations $L_{n,p}(\cdot,k,t)$ of the generalised Jackson operators $L_{n,p}(\cdot,t)$. Generalised Jackson's operators $L_{n,p}$ have been studied by Schurer [56], Schurer and Steutel [57], Lorentz [36] Rathore [49,51] and others.

For any $f \in L_{q,2\pi}$ $(1 \le q < \infty)$, the class of 2π-periodic functions on $\mathbb{R}$ whose restriction on $[-\pi,\pi]$ belongs to $L_q[-\pi,\pi]$, the operator $L_{n,p}(f;t)$ $(n,p \in \mathbb{N})$ is defined by

$$(5.1.1) \quad L_{n,p}(f;t) = \frac{1}{A_{n,p}} \int_{-\pi}^{\pi} \left(\frac{\sin n(\frac{u-t}{2})}{\sin (\frac{u-t}{2})}\right)^{2p} f(u)\, du,$$

where

$$(5.1.2) \quad A_{n,p} = \int_{-\pi}^{\pi} \left(\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}}\right)^{2p} du.$$

The linear combinations $L_n(\cdot,k,t)$ of $L_n(\cdot,t)$ are defined as follows :

Let $d_0,\ldots,d_k$ be k+1 distinct positive integers. Then

$$(5.1.3) \quad L_{n,p}(\cdot,k,t) = \sum_{j=0}^{k} C(j,k)\, L_{d_j n,p}(\cdot,t)$$

where

$$(5.1.4)\qquad C(j,k) = \begin{cases} \sum\limits_{\substack{i=0\\ i\neq j}}^{k} \dfrac{d_j}{d_j-d_i}, & \text{if } k \neq 0 \\ 1, & \text{if } k = 0 \end{cases}$$

Schurer [56] proved the uniform convergence of the sequence $\{L_{n,p}(\cdot,t)\}$ to the functions in $C_{2\pi}$ and he obtained some properties of the operators $L_{n,p}$ for $p = 3,4,5,6$. (we remark here that, the cases $p = 1$ and $p = 2$, correspond to the well known Féjer-and Jackson operators). Schurer and Steutel [57] studied certain constants associated with the approximation by these operators. Lorentz [36] used it to give a proof of Jackson theorems. Rathore [49] made a sup-norm study of certain linear combinations of $L_{n,p}$. Further, he [49] obtained an asymptotic formula for the error in the approximation of derivatives of functions by the corresponding derivatives of the operators.

In this chapter, we prove local direct and inverse theorems for $L_{n,p}(\cdot,k,t)$ of orders $O(\varphi(n^{-1}))$ and $o(\varphi(n^{-1}))$ in L_q-norm $(1 \leq q < \infty)$.

A sectionwise summary of this chapter is as follows : Section 5.2 contains basic results about the operator $L_{n,p}$ and $L_{n,p}(\cdot,k,t)$. Section 5.3 consists of local direct theorem for $L_{n,p}(\cdot,k,t)$ and Sections 5.4 and 5.5, respectively, consist of local $O(\varphi)$ and $o(\varphi)$-inverse theorems for the operators $L_{n,p}(\cdot,k,t)$.

5.2 Basic Results.

Throughout this chapter, we denote $I_j = [a_j, b_j]$, $j=1,2$ and

$$K_n(u) = \frac{1}{A_{n,p}} \left(\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}}\right)^{2p}, \text{where } a_1 < a_2 < b_2 < b_1.$$

Lemma 5.2.1 [49] . Let $m \in \mathbb{N}$ and $2p > m+1$. Then

$$\int_{-\pi}^{\pi} |u|^m K_n(u)\, du \le C_{m,p}\, n^{-m},$$

where $C_{m,p}$ is a constant independent of n.

Lemma 5.2.2 [49] . If $r \le m < 2p-2$, then

$$(5.2.1) \qquad \int_{-\pi}^{\pi} u^r K_n(u)\, du = \sum_{k=r}^{m} \frac{\beta_k}{n^k} + \frac{\varepsilon_n}{n^m},$$

where β_k's are independent of n and $\varepsilon_n \to 0$ as $n \to \infty$.

We can easily observe that if r is odd then the left hand side of (5.2.1) is identically equal to zero.

We prove the convergence of the operators $L_n(\cdot, t)$ for functions in $L_{q,2\pi}$ in

Lemma 5.2.3. Let $f \in L_{q,2\pi}$. Then

$$\|L_{n,p}(f;t) - f(t)\|_{L_q[-\pi,\pi]} \to 0 \text{ as } n \to \infty.$$

Proof. Let $\varepsilon > 0$. Then, there exists $\delta > 0$ such that, whenever $|u| < \delta$,

$$\|f(t+u) - f(t)\|^q_{L_q[-\pi,\pi]} \le \varepsilon.$$

Now,

$$Z = ||L_{n,p}(f;t) - f(t)||^q_{L_q[-\pi,\pi]}$$

$$= \int_{-\pi}^{\pi} \left| \frac{1}{A_{n,p}} \int_{-\pi}^{\pi} \left(\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}}\right)^{2p} \{f(t+u)-f(t)\} du \right|^q dt,$$

which by Jensen's inequality and Fubini's theorem

$$\leq \frac{1}{A_{n,p}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left(\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}}\right)^{2p} |f(t+u)-f(t)|^q \, dt \, du$$

$$\leq \frac{1}{A_{n,p}} \int_{|u|<\delta} \int_{-\pi}^{\pi} |f(t+u)-f(t)|^q \left(\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}}\right)^{2p} dt \, du$$

$$+ \frac{1}{A_{n,p}} \int_{|u|>\delta} \int_{-\pi}^{\pi} |f(t+u)-f(t)|^q \left(\frac{\sin \frac{nu}{2}}{\sin \frac{n}{2}}\right)^{2p} dt \, du$$

$$< \varepsilon/2 + \frac{1}{A_{n,p}} \int_{-\pi}^{\pi} \int_{|u|>\delta} \left(\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}}\right)^{2p} \frac{|u|^m}{\delta^m} \times$$

$$\times |f(t+u) - f(u)|^q \, du \, dt,$$

for any $m > 0$.

Thus, from lemma 5.2.1

$$Z < \varepsilon/2 + \frac{M_2 ||f||^q}{\delta^m} n^{-m}, \text{ whenever } 2p > m+1.$$

Now, choosing n sufficiently large, we get that $Z < \varepsilon$, and hence the result.

The L_p-boundedness is given by the following

$$\leq \frac{1}{A_{n,p}} \int_{-\pi}^{\pi} \left(\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}}\right)^{2p} |u|^{i} \left\{\int_{c}^{d} |f(t+u)|^{q} \frac{|u|^{\ell q}}{\delta^{\ell q}} \, dt\right\}^{1/q} du,$$

noting that for $t \in (c,d)$ and $u \in (-\delta,\delta)$, $t+u \in (a,b)$. Thus

$$Z \leq \frac{1}{A_{n,p}} \frac{1}{\delta^{\ell}} \int_{-\pi}^{\pi} \left(\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}}\right)^{2p} |u|^{i+\ell} \left\{\int_{c}^{d} |f(t+u)|^{q} \, dt\right\}^{1/q} du.$$

Now, using lemma 5.2.1 and the fact that f is periodic, for $2p > i+\ell+1$, we get

$$Z \leq \frac{M}{n^{i+\ell}} ||f||_{L_q[-\pi,\pi]}.$$

Now the result follows, since ℓ is arbitrary but $< 2p-1$.

We close this section by stating a theorem of Rathore [49] about linear combinations.

Theorem 5.2.6. Let f be bounded in the interval $[-\pi+t,\pi+t]$ possessing an r-th derivative at t, $r \leq k < 2p-2$. Then

$$|L_{n,p}(f,k,t) - f(t)| = o(n^{-r})$$

and $(n \to \infty)$.

$$|L_{n,p}(f,r-1,t) - f(t)| = O(n^{-r}).$$

Rathore proved the above theorem for the particular linear combinations taking $d_j = j+1$. But, same proof ramains valid for linear combinations defined in (5.1.3) as well.

5.3 Direct theorem.

We prove a direct theorem in this section which contains

$O(\varphi)$ and $o(\varphi)$ direct theorems as corollaries. The corresponding inverse theorems will be proved in Sections 5.4 and 5.5, respectively.

Theorem 5.3.1. Let $f \in L_{q,2\pi}$ and $p > \frac{k+2}{2}$. Then,

$$||L_{n,p}(f,k,t)-f(t)||_{L_q(I_2)}$$
$$\leq M\{\omega_{k+1}(f,n^{-1},q,I_1) + n^{-(k+1)}||f||_{L_q[-\pi,\pi]}\}.$$

Before proving the theorem, we prove the following results which shall be used in the proof of the theorem.

Lemma 5.3.2. Let $f \in L_{q,2\pi}$ and $i,j \in \mathbb{N}^0$ such that $i+j+2 < 2p$. Then, for any $\ell > 0$ such that $2p > \ell+1$,

$$||L_{n,p}(|u-t|^i|\int_t^u |u-w|^j |f(w)|dw|;t)||_{L_q(I_2)}$$
$$\leq M\{n^{-(i+j+1)}||f||_{L_q(I_1)} + n^{-\ell} ||f||_{L_q[-\pi,\pi]}\}.$$

Proof. Suppose $q > 1$. Let X be the characteristic function of I_1. Then

$$J = ||L_{n,p}(|u-t|^i|\int_t^u |u-w|^j |f(w)|dw|;t)||_{L_q(I_2)}$$
$$\leq ||L_{n,p}(X(u) |u-t|^i |\int_t^u |u-w|^j |f(w)|dw|;t)||_{L_q(I_2)}$$
$$+ ||L_{n,p}((1-X(u))|u-t|^i|\int_t^u |u-w|^j|f(w)|dw|;t)||_{L_q(I_2)}$$

$$= J_1 + J_2, \text{ say.} \tag{5.3.1}$$

First we estimate J_2. Let $\delta = \min(b_1-b_2, a_2-a_1)$. Then clearly $|u-t| \geq \delta$ for all $t \in I_2$ and $\notin I_1$. Hence

$$J_2 \leq \frac{1}{A_{n,p}} \{\int_{a_2}^{b_2} |\int_{-\pi}^{\pi} (\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}})^{2p} (1-X(t+u)) |u|^{i+j} \times$$

$$\times |\int_t^{t+u} |f(w)| dw| du|^q \, dt\}^{1/q}$$

and hence by lemma 0.6.7 and Jensen's inequality, for $m > 0$

$$J_2 \leq \frac{1}{A_{n,p}} \int_{-\pi}^{\pi} (\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}})^{2p} \frac{|u|^{i+j+m}}{\delta^m} \{\int_{a_2}^{b_2} |u|^{q-1} \times$$

$$\times |\int_t^{t+u} |f(w)|^q \, dw| dt\}^{1/q} du.$$

Now, using the periodicity of f, from lemma 5.2.1, we get

$$J_2 \leq \frac{M_1}{n^{i+j+m}} ||f||_{L_q[-\pi,\pi]},$$

whenever $i+j+m-1 < 2p$.

Choosing $\ell = i+j+m$, we get

$$(5.3.2) \qquad J_2 \leq \frac{M_2}{n^{\ell}} ||f||_{L_q[-\pi,\pi]}.$$

Now,

$$J_1 \leq ||L_{n,p}(X(u)|u-t|^{i+j} |\int_t^u |f(w)| dw|;t)||_{L_q(I_2)}$$

$$\leq ||L_{n,p}(X(u)|u-t|^{i+j+1} |H_{|f|}(t)|;t)||_{L_q(I_2)},$$

where $H_{|f|}$ is the Hardy-Littlewood majorant of the function f over the interval I_1.

Now, since $2p > i+j+2$, Jensen's inequality and lemma 5.2.1, imply that

$$(5.3.2) \qquad J_1 \leq \frac{M_3}{n^{i+j+1}} \, ||H|f|||_{L_q(I_1)}$$

$$\leq \frac{M_3}{n^{i+j+1}} \, ||f||_{L_q(I_1)} ,$$

by lemma 0.6.2.

Suppose $q = 1$. Then

$$J \leq \frac{1}{A_{n,p}} \int_{a_2}^{b_2} \int_{-\pi}^{\pi} \left(\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}}\right)^{2p} |u|^{i+j} \left|\int_t^{t+u} |f(w)| dw\right| du \, dt$$

$$= \frac{1}{A_{n,p}} \int_{-\pi}^{\pi} \left(\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}}\right)^{2p} |u|^{i+j} \int_{a_2}^{b_2} \left|\int_t^{t+u} |f(w)| dw\right| dt \, du$$

$$\leq \frac{1}{A_{n,p}} \int_{-\pi}^{\pi} \left(\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}}\right)^{2p} |u|^{i+j} \, |u| \int_{a_2-|u|}^{a_2+|u|} |f(w)| dw \, du$$

$$\leq \frac{1}{A_{n,p}} \int_{|u|<\delta} \left(\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}}\right)^{2p} |u|^{i+j+1} \int_{a_2-|u|}^{b_2+|u|} |f(w)| dw \, du$$

$$+ \frac{1}{A_{n,p}} \int_{|u|\geq\delta} \left(\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}}\right)^{2p} |u|^{i+j+1} \int_{a_2-|u|}^{b_2+|u|} |f(w)| \, dw \, du,$$

where $\delta = \min(a_2-a_1, b_1-b_2)$.

Since $2p > i+j+2$ and $2p > \ell+1$, by lemma 5.2.1,

$$(5.3.3) \qquad J \leq \frac{M_4}{n^{i+j+1}} \, ||f||_{L_1(I_1)} + \frac{M_5}{n^{\ell}} \, ||f||_{L_1[-\pi,\pi]} ,$$

and hence the result follows from (5.3.1-3).

<u>Lemma</u> 5.3.3. Let $f \in L_{q,2\pi}$. If f has $k+1$ derivatives on I_1 with $f^{(k)} \in A.C.(I_1)$, $f^{(k+1)} \in L_q(I_1)$ and $p > \frac{k+2}{2}$, then

$$||L_{n,p}(f,k,t) - f(t)||_{L_q(I_2)}$$

$$\leq \frac{M}{n^{k+1}} \{||f^{(k+1)}||_{L_q(I_1)} + ||f||_{L_q[-\pi,\pi]}\},$$

where M is a constant.

<u>Proof.</u> For $t \in I_2$ and $u \in I_1$,

$$f(u) = \sum_{i=0}^{k} \frac{(u-t)^i}{i!} f^{(i)}(t) + \frac{1}{(k+1)!} \int_t^u (u-w)^k f^{(k+1)}(w)\,dw.$$

Hence, if χ is the characteristic function of I_1, we have

$$L_{n,p}(f,t) = L_{n,p}(\chi f;t) + L_{n,p}((1-\chi)f;t)$$

$$= \sum_{i=0}^{k} \frac{f^{(i)}(t)}{i!} L_{n,p}(\chi(u)\ (u-t)^i;t)$$

$$+ \frac{1}{k!} L_{n,p}(\chi(u) \int_t^u (u-w)^k f^{(k+1)}(w)\ dw;t)$$

$$+ L_{n,p}((1-\chi)f;t).$$

Hence,

$$L_{n,p}(f,k,t)-f(t) = \sum_{i=1}^{k} \{\frac{f^{(i)}(t)}{i!} L_{n,p}(\chi(u)(u-t)^i,k,t)$$

$$+ \frac{1}{k!} L_{n,p}(\chi(u) \int_t^u (u-w)^k f^{(k+1)}(w)\,dw,k,t)$$

$$+ L_{n,p}((1-\chi)f,k,t)$$

(5.3.4) $\quad = J_1 + J_2 + J_3$, say.

Clearly, from lemma 5.2.5 and the boundedness of $C(j,k)$, for any ℓ such that $0 < \ell < 2p-1$, we get

$$(5.3.5) \qquad ||J_3||_{L_q(I_2)} \leq \frac{M_1}{n^{\ell}} ||f||_{L_q[-\pi,\pi]} .$$

Proceeding as in the case of J_1 in lemma 5.3.2 and using the boundedness of $C(j,k)$ again, we get

$$(5.3.6) \qquad ||J_2||_{L_q(I_2)} \leq \frac{M_2}{n^{k+1}} ||f^{(k+1)}||_{L_q(I_1)} .$$

Now,

$$L_{n,p}(X(u)\ (u-t)^i, k, t)$$

$$= L_{n,p}((X(u)-1)(u-t)^i, k, t) + L_{n,p}((u-t)^i, k, t)$$

(5.3.7) $= Z_1 + Z_2$, say.

Proceeding as in lemma 5.2.5, for any ℓ such that $0 < \ell < 2p-1$, we get

$$(5.3.8) \qquad ||Z_1||_{L_q(I_2)} \leq \frac{M_3}{n^{\ell}} ||f||_{L_q[-\pi,\pi]} .$$

Also, by lemma 5.2.2 and the fact that

$\sum_{j=0}^{k} C(j,k)\ d_j^{-m} = 0$, $m = 1,2,\dots,k$, we get

$$(5.3.9) \qquad ||Z_2||_{L_q(I_2)} \leq \frac{M_4}{n^{k+1}} .$$

Hence, combining (5.3.7-9) (taking $\ell = k+1$),

$$||J_1||_{L_q(I_2)} \leq \frac{M_5}{n^{k+1}} \left(\sum_{i=1}^{k} ||f^{(i)}||_{L_q(I_2)} + ||f||_{L_q[-\pi,\pi]} \right)$$

and hence by lemma 0.6.3 , we get

$$(5.3.10) \quad ||J_1||_{L_q(I_2)} \leq \frac{M_6}{n^{k+1}} \{ ||f^{k+1}||_{L_q(I_2)} + ||f||_{L_q(I_2)} \}.$$

The lemma, now, follows from (5.3.4) and the estimates (5.3.5), (5.3.9–10) of J_1, J_2, J_3.

Proof of theorem 5.3.1. We have

$$||L_{n,p}(f,k,t) - f(t)||_{L_q(I_2)} \leq ||L_{n,p}(f-f_{\eta,k+1},k,t)||_{L_q(I_2)}$$

$$+ ||L_{n,p}(f_{\eta,k+1},k,t) - f_{\eta,k+1}(t)||_{L_q(I_2)}$$

$$+ ||f_{\eta,k+1}(t) - f(t)||_{L_q(I_2)} .$$

$$(5.3.11) \qquad = Z_1 + Z_2 + Z_3, \text{ say.}$$

Let X be the characteristic function of $[x,y]$ where $a_1 < x < a_2 < b_2 < y < b_1$, then,

$$Z_1 \leq ||L_{n,p}(X(f-f_{\eta,k+1}),k,t)||_{L_q(I_2)}$$

$$+ ||L_{n,p}((1-X)(f-f_{\eta,k+1}),k,t)||_{L_q(I_2)} .$$

Hence, from lemmas 5.2.4–5 and 0.6.5, we get

$$(5.3.12) \quad Z_1 \leq M_2 \{ \omega_{k+1}(f,\eta,q,I_1) + n^{-(k+1)} ||f||_{L_q[-\pi,\pi]} .$$

Also, from Lemmas 5.3.3 and 0.6.5, we get

(5.3.13) $Z_2 \leq \frac{M_3}{n^{k+1}} \{\eta^{-(k+1)} \omega_{k+1}(f,\eta,q,I_1)+||f||_{L_q[-\pi,\pi]}\}$.

Again, from lemma 0.6.5 , we get

(5.3.14) $Z_3 \leq M_4\, \omega_{k+1}(f,\eta,q,[a_1,b_1])$.

Hence, taking $\eta = n^{-1}$ and combining (5.3.11-14), we get the result.

Corollary 5.3.4. Let $f \in L_{q,2\pi}$, $p > \frac{k+2}{2}$ and $\varphi \in \Phi_{k+1}$. Then,

$$\omega_{k+1}(f,t,q,I_1) = O(\varphi(t)) \quad (t \to 0)$$

implies that

$$||L_{n,p}(f,k,t) - f(t)||_{L_q(I_2)} = O(\varphi(\tfrac{1}{n})) \quad (n \to \infty).$$

Corollary 5.3.5. Let $f \in L_{q,2\pi}$, $p > \frac{k+2}{2}$ and $\varphi \in \Phi_{k+1}$. Then

$$\omega_{k+1}(f,t,q,I_1) = o(\varphi(t)) \quad (t \to 0)$$

implies that

$$||L_{n,p}(f,k,t) - f(t)||_{L_q(I_2)} = o(\varphi(\tfrac{1}{n})) \quad (n \to \infty).$$

5.4 O(φ)-inverse theorem.

In this section we prove the corresponding O(φ)-inverse theorem of corollary 5.3.4.

Theorem 5.4.1. Let $f \in L_{q,2\pi}$, $\varphi \in \Phi_{k+1}$, and $p > \frac{2k+1}{2}$. Then

(5.4.1) $||L_{n,p}(f,k,t)-f(t)||_{L_q(I_1)} = O(\varphi(n^{-1})) \quad (n \to \infty)$

implies that

$$(5.4.2)\quad \omega_{k+1}(f,t,q,I_2) = O(\varphi(t)) \qquad (t \to 0).$$

Before going into the details of the proof, we prove the following

Lemma 5.4.2. Let $f \in L_{q,2\pi}$. Then, for $i \in \mathbb{N}^0$ and $\ell > 0$ such that $2p > i+1, \ell+1$,

$$||L_{n,p}(|u-t|^i |f(u)|;t)||_{L_q(I_2)}$$

$$\leq M\{n^{-i} ||f||_{L_q(I_1)} + n^{-\ell} ||f||_{L_q[-\pi,\pi]}\}.$$

Proof. By generalised Minkowski inequality,

$$J = ||L_{n,p}(|u-t|^i |f(u)|;t)||_{L_q(I_2)}$$

$$\leq \frac{1}{A_{n,p}} \int_{-\pi}^{\pi} \{\int_{a_2}^{b_2} \left(\frac{\sin\frac{nu}{2}}{\sin\frac{u}{2}}\right)^{2pq} \cdot |u|^{iq} |f(t+u)|^q dt\}^{\frac{1}{q}} du$$

$$= \frac{1}{A_{n,p}} \int_{-\pi}^{\pi} \left(\frac{\sin\frac{nu}{2}}{\sin\frac{u}{2}}\right)^{2p} |u|^i \{\int_{a_2}^{b_2} |f(t+u)|^q dt\}^{\frac{1}{q}} du .$$

Hence, if $\delta = \min(a_2-a_1, b_1-b_2)$ then

$$J_1 \leq \frac{1}{A_{n,p}} \int_{-\delta}^{\delta} \left(\frac{\sin\frac{nu}{2}}{\sin\frac{u}{2}}\right)^{2p} \{\int_{a_2}^{b_2} |f(t+u)|^q dt\}^{\frac{1}{q}} du$$

$$+ \frac{1}{A_{n,p}} \int_{\pi\geq|u|>\delta} \left(\frac{\sin\frac{nu}{2}}{\sin\frac{n}{2}}\right)^{2p} |u|^i \left(\int_{a_2}^{b_2} |f(t+u)|^q dt\right)^{\frac{1}{q}} du$$

$$\leq \frac{M_1}{n^i} ||f||_{L_q(I_1)} + \frac{M_2}{n^\ell} ||f||_{L_q[-\pi,\pi]} ,$$

since f is periodic and $2p > \ell+1,\ i+1$. Hence the result.

Lemma 5.4.3. Let $f \in L_{q,2\pi}$ such that supp $f \subset I_2$. Then

$$\|L_{n,p}^{(k+1)}(f;t)\|_{L_q(I_2)} \le M_1\ n^{(k+1)}\ \|f\|_{L_q(I_2)} .$$

Further, if f has k+1 derivatives with $f^{(k)} \in A.C.(I_2)$ and $f^{(k+1)} \in L_q(I_2)$ then

$$\|L_{n,p}^{(k+1)}(f;t)\|_{L_q(I_2)} \le M_2\|f^{(k+1)}\|_{L_q(I_2)} .$$

Proof. We know that $L_{n,p}(f;t)$ is a trigonometric polynomial of degree np-p for any $f \in L_{q,2\pi}$

Hence, by Bernstein inequality [65], since supp $f \subset I_2$, it follows that

$$\|L_{n,p}^{(k+1)}(f;t)\|_{L_q(I_2)} \le (pn)^{k+1}\|L_{n,p}(f;t)\|_{L_q[-\pi,\pi]}$$

and hence the first part follows from lemma 5.2.4.

Since $f^{(k)} \in A.C.(I_2)$ and $f^{(k+1)} \in L_q(I_2)$

$$L_{n,p}^{(k+1)}(f;t) = L_{n,p}(f^{(k+1)};t),$$

and hence the second part of the result follows similarly.

Proof of theorem 5.4.1. Let x_i, y_i, $i = 1,2,3,4$ satisfy $a_1 < x_1 < x_2 < x_3 < x_4 < a_2 < b_2 < y_4 < y_3 < y_2 < y_1 < b_1$. We choose a function $g \in C_o^{k+1}$ such that supp $g \subset (x_3, y_3)$ and $g(t) = 1$ on $[x_4, y_4]$. Writing $fg = \bar{f}$, we have

$$(5.4.3)\quad ||\Delta_\gamma^{k+1}\,\bar{f}(t)||_{L_q[x_3,y_3]}$$

$$\leq ||\Delta_\gamma^{k+1}\,\{\bar{f}(t) - L_{n,p}(\bar{f},k,t)\}||_{L_q[x_3,y_3]}$$

$$+\, ||\Delta_\gamma^{k+1}\, L_{n,p}(\bar{f},k,t)||_{L_q[x_3,y_3]}.$$

By lemma 0.6.1

$$||\Delta_\gamma^{k+1}\, L_{n,p}(\bar{f},k,t)||_{L_q[x_3,y_3]}$$

$$= ||\int_0^\gamma \cdots \int_0^\gamma L_{n,p}^{(k+1)}\,(\bar{f},k,t+\sum_{i=1}^{k+1} z_i)\; dz_1\ldots dz_{k+1}||_{L_q[x_3,y_3]}\,.$$

By repeated applications of Jensen's inequality and Fubini's theorem, for all sufficiently small γ, we have

$$\int_{x_3}^{y_3} |\int_0^\gamma \cdots \int_0^\gamma L_{n,p}^{(k+1)}(\bar{f},k,t+\sum_{i=1}^{k+1} z_i)\; dz_1\ldots dz_{k+1}|^q\, dt$$

$$\leq \gamma^{(k+1)(q-1)} \int_0^\gamma \cdots\int_0^\gamma |L_{n,p}^{(k+1)}(\bar{f},k,t+\sum_{i=1}^{k+1} z_i)|^q\, dt\, dz_1\ldots dz_{k+1}$$

$$\leq \gamma^{(k+1)q}\; ||L_{n,p}^{(k+1)}\,(\bar{f},k,t)||^q_{L_q[x_2,y_2]}\,,$$

since γ is sufficiently small.

Hence,

$$||\Delta_\gamma^{k+1}\, L_{n,p}(\bar{f},k,t)||_{L_q[x_3,y_3]}$$

$$\leq \gamma^{k+1}\; ||L_{n,p}^{(k+1)}\,(\bar{f},k,t)||_{L_q[x_2,y_2]}$$

$$\leq \gamma^{k+1}\; \{||L_{n,p}^{(k=1)}\,(\bar{f}-\bar{f}_{\eta,k+1},k,t)||_{L_q[x_2,y_2]}$$

Hence by lemma 5.4.3, for all sufficiently small $\eta > 0$, we have

$$||\Delta_{\gamma}^{k+1} L_{n,p}(\bar{f},k,t)||_{L_q[x_3,y_3]}$$

$$\leq M_1 \gamma^{k+1} \{n^{k+1} ||\bar{f}-\bar{f}_{\eta,k+1}||_{L_q[x_2,y_2]}$$

$$+ ||\bar{f}^{(k+1)}_{\eta,k+1}||_{L_q[x_2,y_2]}\},$$

which, by lemma 0.6.5, implies that (using the fact that supp $\bar{f} \subset (x_3,y_3)$)

$$(5.4.4) \quad ||\Delta_{\gamma}^{k+1} L_{n,p}(\bar{f},k,t)||_{L_q[x_3,y_3]}$$

$$\leq M_2 \gamma^{k+1}(n^{k+1} + \frac{1}{\eta^{k+1}}) \omega_{k+1}(\bar{f},\eta,q,[x_3,y_3]) .$$

Next we are to show that

$$(5.4.5) \quad ||\Delta_{\gamma}^{k+1} \{\bar{f}(t)-L_{n,p}(\bar{f},k,t)\}||_{L_q[x_3,y_3]} = O(\varphi(n^{-1})) \quad (n \to \infty).$$

After having proved this, we may combine with (5.4.3-4) to get

$$||\Delta_{\gamma}^{k+1} \bar{f}(t)||_{L_q[x_3,y_3]}$$

$$\leq M_3 \{\varphi(n^{-1}) + \gamma^{k+1} (n^{k+1} + \frac{1}{\eta^{k+1}}) \omega_{k+1}(\bar{f},\eta,q[x_3,y_3])\} .$$

Then choosing n such that $n \leq \eta^{-1} < 2n$ and taking supremum over all $\gamma \leq t$, we have

$$\omega_{k+1}(\bar{f},t,q, [x_3,y_3])$$

$$\leq M_4\{\varphi(\eta) + (\frac{t}{\eta})^{k+1} \omega_{k+1}(\bar{f},\eta,q,[x_3,y_3])\},$$

which, by lemma 1.2.2, proves the result.

Now, we prove (5.4.5) by induction as follows :

First, we prove the result for all $\varphi \in \Phi_1$. Later, assuming the validity of the result for all $\varphi \in \Phi_r$ for some r such that $1 \leq r \leq k$, we prove it for $\varphi \in \Phi_{r+1}$.

Let $\varphi \in \Phi_1$. Then

$$\|L_{n,p}(fg,k,t) - (fg)(t)\|_{L_q[x_3,y_3]}$$

$$\leq \|L_{n,p}((f(u)-f(t))\ g(t),k,t)\|_{L_q[x_3,y_3]}$$

$$+ \|L_{n,p}(f(u)(g(u)-g(t)),k,t)\|_{L_q[x_3,y_3]}$$

$$= \|g(t)\{L_{n,p}(f,k,t) - f(t)\}\|_{L_q[x_3,y_3]}$$

$$+ \|L_{n,p}(f(u)(u-t)\ \ g'(\xi),k,t)\|_{L_q[x_3,y_3]},$$

for some ξ lying between u and t. Hence (5.4.1) and lemma 5.4.2 together imply that

$$\|L_{n,p}(fg,k,t) - (fg)(t)\|_{L_q[x_3,y_3]} \leq M_5\ \{\varphi(\tfrac{1}{n}) + \tfrac{1}{n}\} \leq M_6\ \varphi(\tfrac{1}{n}),$$

since $t/\varphi(t) \to 0$ as $t \to 0$.

Thus, we have proved the theorem for all $\varphi \in \Phi_1$.

Now, assume the result for all $\varphi \in \Phi_r$ for some $r \in \mathbb{N}$ such that $1 \leq r \leq k$ and let $\varphi \in \Phi_{r+1}$.

Now,

$$Z = ||L_{n,p}(fg,k,t) - (fg)(t)||_{L_q[x_3,y_3]}$$

$$\leq ||L_{n,p}(f(u)(g(u)-g(t)),k,t)||_{L_q[x_3,y_3]}$$

$$+ ||L_{n,p}((f(u)-f_{\eta,k+1}(u))(g(u)-g(t)),k,t)||_{L_q[x_3,y_3]}$$

$$+ ||L_{n,p}((f_{\eta,k+1}(u)-f_{\eta,k+1}(t))\ (g(u)-g(t),k,t)||_{L_q[x_3,y_3]}$$

$$+ ||L_{n,p}(f_{\eta,k+1}(g(u)-g(t)),k,t)||_{L_q[x_3,y_3]}$$

(5.4.6) $\quad = Z_o + Z_1 + Z_2' + Z_3$, say.

Clearly,

(5.4.7) $\quad Z_o \leq M_7\ \varphi(\frac{1}{n})$.

By theorem 5.2.6 and lemma 0.6.5

(5.4.8) $\quad Z_3 \leq \frac{M_8}{k+1}\ ||f||_{L_q[-\pi,\pi]}$.

Moreover, for some ξ lying between u and t,

$Z_1 = ||L_{n,p}(f(u)-f_{\eta,k+1}(u))\ (u-t)\ g'(\xi),k,t)||_{L_q[x_3,y_3]}$, from

which by lemma 5.4.2 for any ℓ such that $0 < \ell < 2p-1$, we get

$Z_1 \leq M_9\ \{n^{-1}\ ||f-f_{\eta,k+1}||_{L_q[x_2,y_2]} + n^{-\ell}\ ||f||_{L_q[-\pi,\pi]}\}$.

Now, applying lemma 0.6.5.

$$(5.4.9)\quad Z_1 \le M_{10}\,\{n^{-1}\,\omega_{k+1}(f,\eta,q[x_1,y_1] + n^{-\ell}\,||f||_{L_q[-\pi,\pi]}\}.$$

After Taylor's series expansion, we observe that

$$Z_2 \le \frac{1}{k!}\quad ||L_{n,p}((\sum_{i=1}^{k-1} \frac{g^{(i)}(t)}{i!}(u-t)^i) \times$$

$$\times (\int_t^u (u-w)^k\, f_{\eta,k+1}^{(k+1)}(w)\,dw),k,t)||_{L_q[x_3,y_3]}$$

$$+ \frac{1}{(k!)^2}\,||L_{n,p}(g^{(k)}(\xi)(u-t)^k \times$$

$$\times \int_t^u (u-w)^k\, f_{\eta,k+1}^{(k+1)}(w)\,dw,k,t)||_{L_q[x_3,y_3]}$$

$$+ \sum_{i=1}^{k}\sum_{j=1}^{k-1} \frac{1}{i!\,j!}||f_{\eta,k+1}^{(i)}(t)g^{(j)}(t)L_{n,p}((u-t)^{i+j},k,t)||_{L_q[x_3,y_3]}$$

$$+ \frac{1}{k!}\,\{\sum_{i=1}^{k} \frac{1}{i!}\,||f_{\eta,k+1}^{(i)}(t)\;L_{n,p}((u-t)^{k+i}\,g^{(k)}(\xi),k,t)||_{L_q[x_3,y_3]}$$

$$(5.4.10)\qquad = R_1 + R_2 + R_3 + R_4,\ \text{say}.$$

By lemma 5.3.2, for any ℓ such that $0 < \ell < 2p-1$, we get

$$(5.4.11)\quad R_1 \le M_{10}\,\{\sum_{i=1}^{k-1} n^{-(k+i)}\,||f_{\eta,k+1}^{(k+1)}||_{L_q[x_1,y_1]}$$

$$+ n^{-\ell}\,||f_{\eta,k+1}^{(k+1)}||_{L_q[-\pi,\pi]}\}$$

and

$$(5.4.12)\quad R_3 \le M_{11}\,\{n^{-(2k+1)}\,||f_{\eta,k+1}^{(k+1)}||_{L_q[x_1,y_1]}$$

$$+ n^{-\ell}\,||f^{(k+1)}_{\eta,k+1}||_{L_q[-\pi,\pi]}\}.$$

It follows from lemma 5.2.2 and the fact that $\sum_{j=0}^{k} C(j,k)\, d_j^{-m} = 0,\ m = 1,2,\dots,k,$ that

$$R_3 \le \frac{M_{12}}{n^{k+1}} \Big(\sum_{i=1}^{k} \|f_{\eta,k+1}^{(i)}\|_{L_q[x_3,y_3]} \Big).$$

Hence by lemma 0.6.3

$$(5.4.13)\quad R_3 \le \frac{M_{13}}{n^{k+1}} \{\|f_{\eta,k+1}^{(k)}\|_{L_q[x_3,y_3]} + \|f_{\eta,k+1}\|_{L_q[x_3,y_3]}\}$$

and

$$(5.4.14)\quad R_4 \le \frac{M_{14}}{n^{k+1}} \{\|f_{\eta,k+1}^{(k)}\|_{L_q[x_3,y_3]} + \|f_{\eta,k+1}\|_{L_q[x_3,y_3]}\}.$$

Choosing n such that $n < \eta^{-1} \le 2n$, it follows from (5.4.11-14) and lemma 0.6.5 , that

$$(5.4.15)\quad R_1, R_2 \le M_{16}\{n^{-1}\, \omega_k(f,n^{-1},q,[x_1,y_1]) + n^{-(k+1)} \|f\|_{L_q[-\pi,\pi]}\}$$

and

$$(5.4.16)\quad R_3, R_4 \le M_{17}\{n^{-1}\, \omega_{k+1}(f,n^{-1},q[x_1,y_1]) + n^{-(k+1)} \|f\|_{L_q[-\pi,\pi]}\}.$$

Now, clearly $\frac{\varphi(t)}{t} \in \Phi_r$ and hence by induction hypothesis, we have

$$\omega_{k+1}(f,t,q,[x_1,y_1]) = O\Big(\frac{\varphi(t)}{t}\Big) \qquad (t \to 0)$$

and hence, by lemma 1.2.4, we get

$$\omega_k(f,t,q,[x_1,y_1]) = O\Big(\frac{\varphi(t)}{t}\Big) \qquad (t \to 0).$$

Thus, for j = k, k+1,

(5.4.17) $\quad n^{-1}\,\omega_j(f,n^{-1},q,[x_1,y_1]) = O(\bar{\varphi}(n^{-1})) \qquad (n \to \infty).$

Also since $\varphi \in \Phi_{r+1} \subset \Phi_{k+1}$,

(5.4.18) $\quad \dfrac{n^{-(k+1)}}{\varphi(n^{-1})} \to 0$ as $n \to \infty$.

Hence, in view of (5.4.17-18), combining (5.4.7-10) and (5.4.15-16), we get

$$Z = Z_o + Z_1 + Z_2 + Z_3 \leq M_{19}\,\bar{\varphi}(n^{-1})$$

and hence the theorem is proved.

<u>Corollary</u> 5.4.4. Let $f \in L_{q,2\pi}$ and $1 \leq r \leq k+1$ and let $\varphi \in \Phi_r$. Then

$$||L_{n,p}(f,k,t) - f(t)||_{L_q(I_1)} = O(\bar{\varphi}(n^{-1}))$$

implies that

$$\omega_r(f,t,q,I_1) = O(\varphi(t)) \qquad (t \to 0).$$

5.5 <u>$o(\varphi)$-inverse theorem</u>.

In section 5.3 we proved a $\bar{o}(\varphi)$-direct theorem (corollary 5.3.4). In this section, we prove the corresponding $o(\varphi)$-inverse theorem.

<u>Theorem</u> 5.5.1. Let $f \in L_{q,2\pi}$, $p > \frac{2k+1}{2}$ and $\varphi \in \Phi_{k+1}$. Then

(5.5.1) $\quad ||L_{n,p}(f,k,t) - f(t)||_{L_q(I_1)} = \bar{o}(\varphi(n^{-1})) \qquad (n \to \infty)$

implies that

(5.5.2) $\quad \omega_{k+1}(f,t,q,I_2) = \bar{o}(\varphi(t)) \qquad (t \to 0).$

Proof. Choosing $x_i, y_i, g, \bar{f}$ as in the proof of theorem 5.4.1 and proceeding as in the proof of the same theorem, we get

$$(5.5.3)\quad \|\Delta_{\gamma}^{k+1} L_{n,p}(\bar{f},k,t)\|_{L_q[x_3,y_3]}$$

$$\le M_1\ \gamma^{k+1}\ (n^{k+1} + \frac{1}{\eta^{k+1}})\ \omega_{k+1}(\bar{f},\eta,q,[x_3,y_3]).$$

Next we have to show that

$$(5.5.4)\quad \|\Delta_{\gamma}^{k+1} \{\bar{f}(t) - L_{n,p}(\bar{f},k,t)\}\|_{L_q[x_3,y_3]} = o(\varphi(n^{-1}))\ (n \to \infty).$$

Now define $\psi(x)$ as follows :

$$(5.5.5)\quad \psi(x) = \begin{cases} \|\Delta_{\gamma}^{k+1}\{\bar{f}(t)-L_{n,p}(\bar{f},k,t)\}\|_{L_q[x_3,y_3]} & \text{if } x = n^{-1} \\ \psi(n^{-1}) & \text{if } x \in ((n+1)^{-1},n^{-1}). \end{cases}$$

Clearly $\psi(x) = o(\varphi(x))$ $(x \to 0)$ and hence, after having proved (5.5.4) combining (5.5.3-5), we get

$$\|\Delta_{\gamma}^{k+1}\ \bar{f}(t)\|_{L_q[x_3,y_3]}$$

$$\le M_2\ \{\psi(n^{-1}) + \gamma^{k+1}(n^{k+1} + \frac{1}{\eta^{k+1}})\ \omega_{k+1}(\bar{f},\eta,q,[x_3,y_3])\}.$$

Now, choosing n such that $n \le \eta^{-1} < n+1$ and taking supremum over all $\gamma \le t$, we have

$$\omega_{k+1}(\bar{f},t,q,[x_3,y_3])$$

$$\le M_4\ \{\psi(\eta) + (\frac{t}{\eta})^{k+1}\ \omega_{k+1}(f,\eta,q,[x_3,y_3])\}.$$

Now, conclusion (5.5.2) follows from lemma 1.2.3.

Thus, we are left to show that

(5.5.6) $\|fg(t) - L_{n,p}(fg,k,t)\|_{L_q[x_3,y_3]} = o(\varphi(n^{-1}))$ $(n \to \infty)$.

As in the proof of theorem 5.4.1, we prove this by induction as follows : First, we prove the theorem for all $\varphi \in \Phi_1$. Next, assuming the result of the theorem for all $\varphi \in \Phi_r$ for some r such that $1 \leq r \leq k$, we prove the result for all $\varphi \in \Phi_{r+1}$.

Let $\varphi \in \Phi_1$. Then, from hypothesis and lemma 5.4.2, we have

$$\|(fg)(t) - L_{n,p}(fg,k,t)\|$$

$$\leq \|L_{n,p}((f(u)-f(t))\ g(t),k,t)\|_{L_q[x_3,y_3]}$$

$$+ \|L_{n,p}(f(u)(g(u)-g(t)),k,t)\|_{L_q[x_3,y_3]}$$

$$\leq M_3\ \{\varphi(\tfrac{1}{n}) + \tfrac{1}{n}\}$$

and hence (5.5.6) is proved, for all $\varphi \in \Phi_1$. Thus, the theorem is proved for all $\varphi \in \Phi_1$.

Now, assume that the theorem holds for all $\varphi \in \Phi_r$ for some r such that $1 \leq r \leq k$ and let $\varphi \in \Phi_{r+1}$.

Then, clearly

$$\frac{\varphi(t)}{t} \in \Phi_r.$$

Hence, by induction hypothesis, we have

$$\omega_{k+1}(f,t,q,[x_1,y_1]) = o(\frac{\varphi(t)}{t}) \quad (t \to 0),$$

ıich, by lemma 1.2.5, implies that

$$\omega_k(f,t,q,[x_1,y_1]) = o(\frac{\varphi(t)}{t}) \quad (t \to 0).$$

ıus, we get, for $j = k, k+1$,

$$\text{.5.7)} \qquad n^{-1}\,\omega_j(f, \tfrac{1}{n}, q,[x_1,y_1]) = o(\varphi(\tfrac{1}{n})) \quad (n \to \infty).$$

so, as $\varphi \varepsilon \Phi_{k+1}$, we have

$$\text{.5.8)} \qquad n^{-(k+1)}/\varphi(n^{-1}) \to 0 \quad \text{as } n \to \infty.$$

Now, proceeding again as in the proof of theorem 5.4.1, get

$$L_{n,p}(fg,k,t)-(fg)(t)||_{L_q[x_3,y_3]} \leq Z_0 + Z_1 + Z_2 + Z_3 .$$

Clearly, by hypothesis (5.5.1)

$$Z_0 = o(\varphi(\tfrac{1}{n})) \qquad (n \to \infty).$$

so, from theorem 5.2.6 and lemma 0.6.5 and (5.5.8), we get

$$Z_3 = o(\varphi(\tfrac{1}{n})) \qquad (n \to \infty).$$

Z_1, Z_2 were estimated in the proof of theorem 5.4.1 as

$$Z_1 \leq M_5\{n^{-1}\,\omega_{k+1}(f,\eta,q,[x_1,y_1]) + n^{-\ell}\,||f||_{L_q[-\pi,\pi]}\}$$

$$_2 \leq R_1 + R_2 + R_3 + R_4, \text{ where}$$

$$M_6\{n^{-1}\,\omega_k(f,n^{-1},q,[x_1,y_1]) + n^{-(k+1)}\,||f||_{L_q[-\pi,\pi]}\}$$

and

$$R_3, R_4 \leq M_8\{n^{-1}\omega_{k+1}(f, n^{-1}, q, [x_1, y_1]) + n^{-(k+1)}||f||_{L_q[-\pi,\pi]}\}.$$

Thus, from (5.5.7) we see that

$$Z_1, Z_2 = o(\varphi(\frac{1}{n}))$$

and hence the result.

We conclude this chapter with the following two corollaries :

Corollary 5.5.2. Let $f \varepsilon L_{q,2\pi}$, $p > \frac{2k+1}{2}$ and $\varphi \varepsilon \Phi_r$ for some r such that $1 \leq r \leq k+1$. Then

$$||L_{n,p}(f,k,t) - f(t)||_{L_q(I_1)} = o(\varphi(\frac{1}{n}))$$

implies that

$$\omega_{k+1}(f,t,q,I_2) = o(\varphi(t)) \quad (t \to 0).$$

Corollary 5.5.3 (Global φ-inverse theorems). Let $f \varepsilon L_{q,2\pi}$ and $p > \frac{2k+1}{2}$. Let $\varphi \varepsilon \Phi_{k+1}$. Then

$$||L_{n,p}(f,k,t) - f(t)||_{L_q[-\pi,\pi]} = O(\varphi(\frac{1}{n})) \quad (n \to \infty)$$

if and only if

$$\omega_{k+1}(f,t,q,[-\pi,\pi]) = O(\varphi(t)) \quad (t \to 0).$$

Moreover, above statement remains valid if O is replaced by o everywhere.

Proof. Considering an interval $[a_1, b_1]$ where $a_1 < -\pi < \pi < b_1$ and using the periodicity of f, we get the result from the earlier $O(\varphi)$ and $o(\varphi)$-inverse theorems (Theorems 5.4.1 and 5.5.1).

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